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| RESEARCH ARTICLE

Low-frequency skin effect in Aluminum conductors: Theory, simulation, and experiment

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| ABSTRACT

Electromagnetic field propagation in conductive media and the resulting skin effect are basis to the analysis of power systems, electromagnetic shielding, and signal transmission. Although classical electromagnetic theory accurately describes skin depth and its dependence on material properties and frequency, experimental investigation at very low frequencies remains challenging. In good conductors such as aluminum, the skin depth at power-line frequencies is comparable to typical conductor dimensions, leading to weak attenuation and an almost uniform current distribution. In this study, the skin depth of aluminum at 50 Hz was investigated using a combined theoretical, numerical, and experimental approach. Analytical expressions for the attenuation constant and skin depth were derived from Maxwell's equations for a linear, isotropic conducting medium, yielding a theoretical skin depth of approximately 11.60 mm at 50 Hz. Numerical simulations were performed in the frequency domain using COMSOL Multiphysics 5.5 to investigate the current density distribution and the frequency-dependent behavior of the skin effect, showing a nearly uniform current distribution in a 10 mm-radius aluminum wire at 50 Hz and a prominent skin effect at higher frequencies. Experimental measurements were carried out using aluminum sheets of varying thickness, and the attenuation of the electric field amplitude was analyzed to estimate the experimental skin depth. The experimentally obtained value, approximately 36.6 mm, was larger than the theoretical prediction. This discrepancy is attributed to finite sample geometry, edge effects, measurement uncertainty, and deviations from ideal material assumptions. The results highlight the limitations of low-frequency skin depth measurements and provide insight into the factors governing electromagnetic field behavior in conductive media under practical laboratory conditions.

| KEYWORDS

Skin effect; Aluminum; Low frequency; Simulation.

| ARTICLE INFORMATION

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1. Introduction

Propagation of electromagnetic fields in a conductive medium is one of the fundamental topics in classical electromagnetism and plays a significant role in the analysis and design of power systems, transmission lines, electromagnetic protection, and signal management in communications[1-4]. The behavior of electromagnetic fields in conductors is directly influenced by the electrical properties of the material and the frequency of the applied field[5]. One of the key parameters that describes this behavior is the skin depth, which determines the degree of penetration of the field and electrical current into the conductor[4-6].

From a theoretical perspective, skin depth is precisely derived from Maxwell's equations, and its dependence on frequency, electrical conductivity, and magnetic permeability of the material is well-known[7]. These classical models, especially at high frequencies, show good agreement with experimental results and are widely used in scientific and engineering issues[8]. However, at very low frequencies, especially in the power system frequency range such as 50 Hz, experimental investigation of skin depth is associated with significant challenges due to the large penetration depth in typical conductors[8,9]. At low frequencies, the skin depth in good conductors such as aluminum becomes large, allowing the electromagnetic field to penetrate a significant portion of the conductor's cross-section[10]. Under these conditions, the current distribution is nearly uniform, and the effects of skin effect are very weak. Consequently, direct measurement of the skin depth using conventional laboratory methods is difficult, and variation in parameters such as AC resistance or field attenuation are often on the same order of magnitude as the uncertainty of the measuring instrument and environmental noise[8,11]. In addition to the inherent limitations arising from the low frequency, the ideal assumptions used in theoretical models are not always fully valid in experimental conditions. In theoretical calculations, the electrical conductivity of the material is assumed to be constant and uniform[9], whereas in practice, factors such as the purity of the aluminum, temperature and physical structure of the sample can cause changes in the effective electrical properties[12,13]. As a consequence, the experimental results can deviate from the values predicted by theory. Therefore, the observed discrepancy between theoretical and experimental values of skin depth at low frequencies cannot be regarded merely as experimental error or inefficiency of theoretical models. Rather, this discrepancy reflects the practical limitations of measurement and the physical complexities of the laboratory environment. Investigating this discrepancy and identifying the factors influencing it can lead to a deeper understanding of the behavior of electromagnetic fields in conductors under low-frequency conditions and reduce the gap between theoretical descriptions and experimental observations.

Accordingly, the main objective of this study is to investigate the skin depth of aluminum at a frequency of 50 Hz using both theoretical and experimental approaches, and to analyze the reasons of discrepancy between the results obtained from these two methods. The results of this research can provide a clear framework for the interpretation of low-frequency experiments related to the skin effect and help improve the conceptual understanding of this phenomenon for researchers and students.

2. Methods

This study employs both quantitative and qualitative methods, integrating experimental measurements with theoretical formulations. Experimentally, the dependence of electromagnetic wave amplitude on the thickness of an aluminum medium was investigated at 50 Hz frequency. The theoretical analysis is based on Maxwell's equations, which are used to describe the skin depth phenomenon and derive its related expression. Furthermore, the skin effect was simulated using COMSOL Multiphysics 5.5 at different frequencies including 50 Hz to visualize the distribution of current over the cross section of a wire.

For the experimental investigation, a 12 V power supply, aluminum (Al) sheets of varying thicknesses, a cathode ray oscilloscope (CRO), and an amplifier were used. By inserting aluminum sheets of different thicknesses into the setup, the amplitude of the electric field (E) was measured and recorded. Mathematically, electric filed component of an electromagnetic wave inside a conductor is given by

$$E(z) = E_0\, e^{-\kappa z} \tag{1}$$

where E_0 is the field amplitude at the surface and κ is the attenuation constant[4]. The skin depth δ is defined as the depth at which the field amplitude decreases to E_0/e[9]. Accordingly, by setting z=δ and $E = \frac{E_0}{e}$ in Eq. 1, the skin depth is obtained as

$$\delta = \frac{1}{\kappa} \tag{2}$$

Taking the natural logarithm of Eq. (1) yields

$$\ln E = -\kappa z + \ln E_0 \tag{3}$$

By comparison with the linear equation $y = mx + c$, the slope of the linear relation is equal to -κ. Therefore, the experimental skin depth was determined from,

$$\delta = \frac{1}{|slope|} \tag{4}$$

Accordingly, the natural logarithm of the measured electric field amplitude (ln E) was calculated for each recorded data point and plotted as a function of aluminum thickness (z) using Origin Project software. The resulting data were fitted with a linear function, and the slope of the fitted line was used to determine the skin depth.

3. Results and Discussion

The transmission behavior of electromagnetic waves in a conductive medium, specifically aluminum plates, was examined through combined theoretical and experimental approaches.

3.1 Theoretical Approach

The Maxwell's equations for linear, isotropic conducting medium are given by

$$\nabla.\vec{E} = \frac{\rho_f}{\epsilon} \;; \qquad \nabla.\vec{B} = 0 \tag{5}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \;; \qquad \nabla \times \vec{B} = \mu\sigma\vec{E} + \mu\epsilon\frac{\partial \vec{E}}{\partial t} \tag{6}$$

Where, μ is the magnetic permeability and σ is the electrical conductivity of the material[14]. By taking the curl of the relevant Maxwell's equations in Eq. (6), the wave equations for the electric and magnetic fields are obtained as

$$\nabla^2\vec{E} = \mu\epsilon\frac{\partial^2 \vec{E}}{\partial t^2} + \mu\sigma\frac{\partial \vec{E}}{\partial t} \quad , \quad \nabla^2\vec{B} = \mu\epsilon\frac{\partial^2 \vec{B}}{\partial t^2} + \mu\sigma\frac{\partial \vec{B}}{\partial t} \tag{7}$$

The solutions of these equations may be expressed in the form of plane electromagnetic waves propagating in the z-direction

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(\tilde{k}z-\omega t)} \quad , \quad \tilde{B}(z,t) = \tilde{B}_0 e^{i(\tilde{k}z-\omega t)} \tag{8}$$

where the complex wave number ($\tilde{k}$) is written as

$$\tilde{k} = k + i\kappa \tag{9}$$

The substitution of Eq. (8) in Eq. (7) yields

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \tag{10}$$

Equating the real and imaginary parts of Eqs. (9) and (10) leads to

$$k^2 - \kappa^2 = \mu\epsilon\omega^2 \quad , \quad 2k\kappa = \mu\sigma\omega \tag{11}$$

By eliminating κ from the above equations, an expression for k was obtained as follows:

$$k^4 - \left(\frac{\mu\sigma\omega}{2}\right)^2 = \mu\epsilon\omega^2 k^2 \tag{12}$$

This is a quadratic equation. Solving for k^2 and subsequently for k, and then determining κ form Eq. (11), yields

$$k \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{\frac{1}{2}} \;,\; \kappa \equiv \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{\frac{1}{2}} \tag{13}$$

Now the complex wavenumber ($\tilde{k}$) from Eq. 9 implies that

$$\tilde{E}(z,t) = \tilde{E}_0 e^{-\kappa z}\, e^{i(kz-\omega t)} \quad , \quad \tilde{B}(z,t) = \tilde{B}_0 e^{-\kappa z} e^{i(kz-\omega t)} \tag{14}$$

The exponential factor $e^{-\kappa z}$ represents attenuation of the wave amplitude with depth[15]. The distance over which the field amplitude decreases to 1/e of its surface value is defined as the **skin depth** δ. For weak conductors ($\sigma \ll \omega\epsilon$), Eq. (13) reduced to

$$k \cong \omega\sqrt{\epsilon\mu}\,, \qquad \kappa \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\epsilon}}$$

For good conductors ($\sigma \gg \omega\epsilon$) which applies to metals such as aluminum, Eq. (13) simplifies to

$$k \approx \kappa \equiv \sqrt{\frac{\omega\sigma\mu}{2}}$$

Thus, considering Eq. (2), the theoretical skin depth of a good conductor is given by

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu}}$$

Where, ω=2πf is the angular frequency, μ is the magnetic permeability of the material, and σ is the electrical conductivity[6]. Since, specific conductivity of aluminum at room temperature is $3.70 \times 10^{7}\ \Omega^{-1} m^{-1}$ and the magnetic permeability in vacuum is $4\pi \times 10^{-7} N/Amp^{2}$[16]; therefore, after substituting these values into the skin depth equation yields a theoretical skin depth relation dependent on frequency

$$\delta = \frac{81.97}{\sqrt{f}}\ mm$$

where f is in terms of Hz and should be unitless in the above equation because its unit already has been cancelled. Since the frequency in this investigation is 50 Hz, therefore the above equation gives a skin depth of approximately, $11.60\ mm$.

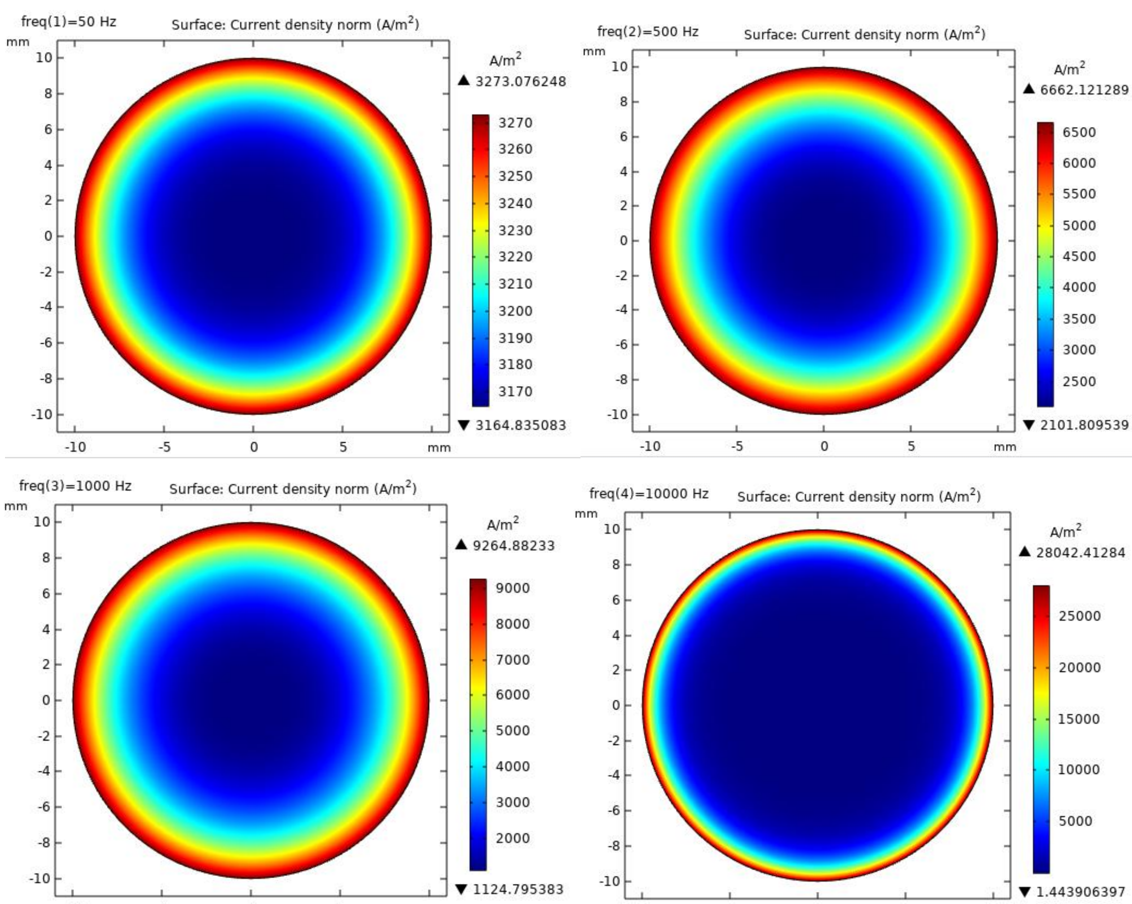
This result indicates that, at power-line frequency, electromagnetic fields penetrate relatively deeply into aluminum. The calculated theoretical skin depth is significantly larger than the typical thickness of aluminum conductors used in practical and laboratory applications. This confirms that, at low frequencies, aluminum behaves as a weakly attenuating conductor, and the current density remains nearly uniform across much of the conductor cross section[10,17].

The large value of skin depth at 50 Hz arises basically from the low angular frequency of the applied field. Since skin depth is inversely proportional to the square root of frequency, substantial field confinement near the surface only occurs at much higher frequencies[18]. Consequently, at power-line frequencies, the skin effect in aluminum is relatively weak, and its impact on current distribution is limited.

The theoretical calculation assumes an ideal conductor with uniform electrical conductivity, constant temperature, and infinite extent. Under these assumptions, the electromagnetic field inside the conductor decays exponentially with depth, and the attenuation constant is uniquely defined by the material parameters. However, in practical situations, deviations from this ideal behavior are expected due to finite geometry, temperature variation, and material imperfections[19].

3.2 COMSOL Simulations

The skin effect in aluminum was simulated using COMSOL Multiphysics 5.5 with the Magnetic Fields (mf) interface from the AC/DC Module, employing a frequency-domain study (Fig. 1).



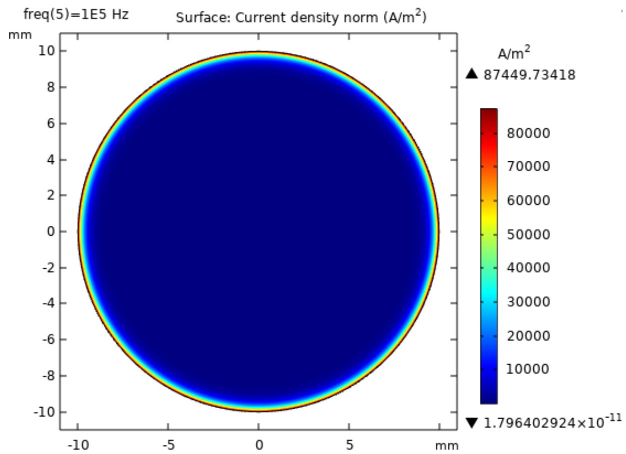


Fig. 1 Simulated skin effect in aluminum at frequencies of 50 Hz, 500 Hz, 1 kHz, 10 kHz, and 100 kHz, showing the radial distribution of current density in the conductor cross-section

The conductor geometry was modeled as a circular cross-section with a radius of 10 mm. The electrical conductivity and relative permeability of aluminum were set to 3.774×10^7 S/m and $\mu_r = 1$, respectively. A single-turn coil carrying an AC current of 1 A was used to excite the conductor. To accurately resolve the skin effect near the conductor surface, a boundary layer mesh consisting of eight layers, a stretching factor of 1.2, and a first-layer thickness of $\delta/5$ was applied, in combination with a free triangular mesh in the remaining domain.

According to theory, skin depth decreases with frequency. The simulations qualitatively illustrate this effect through current density distribution. The frequency-domain simulations of current density in a 10 mm-radius aluminum wire illustrate the classic behavior of the skin effect. At 50 Hz, the current distribution is nearly uniform across the wire cross-section, consistent with theoretical expectations for low-frequency conduction in conductors of relatively large diameter. As the frequency increases to 500 Hz and 1 kHz, a gradual shift of the current toward the wire surface becomes apparent. This indicates the onset of the skin effect, where the interior of the conductor carries less current due to the electromagnetic self-induction opposing the flow in the core.

At higher frequencies, 10 kHz and 100 kHz, the simulations show a prominent concentration of current near the conductor surface, with the interior almost free of significant current flow. These results qualitatively confirm the inverse square-root relationship between skin depth and frequency, demonstrating that higher frequencies confine the current to progressively thinner layers near the surface. While these simulations provide only a visual illustration without quantitative calculation of skin depth, they effectively highlight the frequency-dependent redistribution of current within the conductor and serve as a bridge between theoretical predictions and potential experimental observations.

3.3 Experimental Approach

Based on the experimental measurements, the wave amplitude was recorded for aluminum sheets of varying thicknesses. Using Eq. (4), the corresponding values of $\ln E$ and $\ln E/E_0$ were calculated for each thickness, as summarized in Table 1.

Table 1. Measured wave amplitude for aluminum sheets of varying thicknesses and the corresponding calculated values of $\ln E$ and $\ln E/E_0$.

No	Thickness of plates (mm)	Amplitude (E) (y*0.05 mV)	ln E	ln(E/E0)
1	0	3.6	1.2809	0
2	6.005	3.4	1.2238	-0.0572
3	8.009	3.2	1.1632	-0.1178
4	10.011	3.0	1.0986	-0.1823
5	12.010	2.8	1.0296	-0.2513

No	**Thickness of plates (mm)**	**Amplitude (E) (y*0.05 mV)**	**ln E**	**ln(E/E0)**
6	14.012	2.6	0.9555	-0.3254
7	16.025	2.6	0.9555	-0.3254
8	18.022	2.4	0.8755	-0.4055
9	19.024	2.4	0.8755	-0.4055
10	24.032	2.2	0.7885	-0.4925

The measured amplitude of the electric field decreases as the thickness of the aluminum sheets increases, confirming the expected attenuation of electromagnetic waves within a conducting medium[20]. The calculated values of *ln* E/E_0 show a nearly linear decrease with increasing thickness, consistent with the theoretical model of exponential decay (Eq. 1). Small deviations from perfect linearity are observed at thicknesses above 14 mm, likely due to measurement uncertainties, finite sample size, and instrument resolution. These data were subsequently used to construct a plot of *ln* E/E_0 versus thickness, from which the slope was extracted to determine the experimental skin depth (Fig. 2).

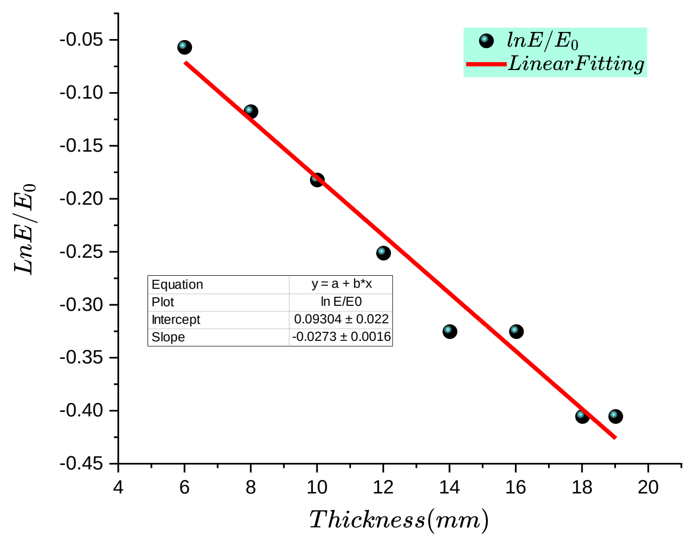


Fig.2 Plot of $\ln\frac{E}{E_0}$ versus aluminum sheet thickness, showing the exponential attenuation of the electric field with increasing thickness.

A linear fit was applied to the *ln* E/E_0 versus thickness data, and the slope of the fitted line was used to evaluate the experimental skin depth. The absolute value of the slope obtained 0.0273, which corresponds to an experimental skin depth of approximately **36.63 mm** using Eq. (4). This value is notably larger than the theoretical skin depth of aluminum at 50 Hz, which was around 11.6 mm based on standard material parameters. The discrepancy can be attributed to several factors. First, the experimental setup involves finite aluminum sheets, whereas the theoretical model assumes an ideal, infinitely large conductor[9]. Edge effects and non-uniform field distribution near the boundaries reduce the measured attenuation, effectively increasing the apparent skin depth[21]. Second, measurement uncertainties in field amplitude and instrument resolution, especially at low frequencies where attenuation is weak, contribute to the difference. Third, variations in material properties may further influence the experimental result. Despite these deviations, the observed trend exponential decay of amplitude with thickness confirms the remarkable theoretical prediction of field attenuation in a conducting medium and validates the methodology for estimating skin depth experimentally.

4. Conclusion

This study investigated the skin effect in aluminum at low frequency through a combination of theoretical analysis, numerical simulation, and experimental measurement. Starting from Maxwell's equations, the classical expression for skin depth in a good conductor was derived, and a theoretical value of approximately 11.6 mm was obtained for aluminum at the power-line frequency of 50 Hz. This large skin depth indicates weak attenuation of electromagnetic fields and an almost uniform current distribution across the conductor cross-section, consistent with established electromagnetic theory.

Frequency-domain simulations using COMSOL Multiphysics 5.5 further illustrated the behavior of current density in aluminum conductors. The numerical results confirmed that, at low frequencies, the skin effect is weak, while increasing frequency leads to progressive confinement of current near the conductor surface. These simulations provide a useful visualization of the theoretical predictions and support the analytical results.

Experimental measurements of electromagnetic wave attenuation in aluminum sheets of increasing thickness revealed an approximately exponential decay of field amplitude, enabling an experimental skin depth of about 36.6 mm at 50 Hz to be extracted. The difference between the experimental and theoretical values cannot be attributed merely to experimental error. Instead, it reflects the combined influence of finite sample dimensions, edge effects, measurement limitations, and variations in material properties that are not considered in idealized theoretical models.

Overall, the results demonstrate that discrepancies between theoretical and experimental skin depth values at low frequencies are an inherent consequence of practical measurement conditions rather than a failure of classical electromagnetic theory. This work provides a transparent framework for interpreting low-frequency skin effect experiments and contributes to a more realistic understanding of electromagnetic field propagation in conductive media, particularly for educational and laboratory-scale investigations.

Autor contributions

The concept of the study was developed by Sayed A. S. Mosamem, who also conducted the experimental work, plotted the data using Origin software, simulated the skin effect in COMSOL Multiphysics, drafted the manuscript. Enayatullah Samim contributed by preparing tables, performing necessary calculations, and reviewing the relevant literature. Mujibullah Mahboob contributed by finalizing the manuscript through fixing the paragraphs, validation of citations and adjusting the structure of manuscript according to journal recommendations.

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