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**| RESEARCH ARTICLE**

# Line Graphs of Zero-Divisor Graphs of Finite Free Semilattices

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**| ABSTRACT**

Let $SL_n$be the finite free semilattice on $X_n = \{1,2,\ldots,n\}$ and $\Gamma_n$ be the zero-divisor graph of $SL_n$. Moreover, let $\mathrm{L}(\Gamma_n)$ be the line graph of $\Gamma_n$. In this paper, we find the cardinality of the vertex set of $\mathrm{L}(\Gamma_n)$, diameter, maximum degree, minimum degree, girth, clique number, chromatic number, and degrees of all vertices in $\mathrm{L}(\Gamma_n)$ for $n \geq 3$.

**| KEYWORDS**

Finite free semilattice, line graph, diameter, clique number.

**| ARTICLE INFORMATION**

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## 1. Introduction

In 1998, Beck defined the zero-divisor graph of a commutative ring [2], the zero element of ring was a vertex in Beck's definiton. Then, Anderson and Livingston redefined the zero-divisor graph of a commutative ring and this is the standart zero-divisor graph of a commutative ring [1]. In a similar way, Demeyer et al. defined the zero-divisor graph of a commutative semigroup with zero [3]. Let $S$ be a commutative semigroup with zero, 0 be the zero element of $S$ and $Z(S)$ be the set of zero-divisors of $S$. The zero-divisor graph of $S$ is denoted by $\Gamma(S)$ and it is an undirected graph with vertex set $Z(S)^* = Z(S)\backslash\{0\}$ and distinct two vertices $\alpha$ and $\beta$ are adjacent vertices in $\Gamma(S)$ if and only if $\alpha\beta = 0$.

Let $n \in Z^+$ and $X_n = \{1,2,\ldots,n\}$. Let $SL_n$ be the set consisting of all subsets of $X_n$ except the empty set. Let $A.B = A \cup B$ for all $A, B \in SL_n$. Then, $SL_n$ is a commutative semigroup and it is called the free semilattice on $X_n$. The zero-divisor graph of $SL_n$ is denoted by $\Gamma(SL_n)$. In 2016, many properties of $\Gamma(SL_n)$ were investigated [6]. Let $G$ be a graph and $L(G)$ be the line graph of $G$. In this paper, some basic properties of $L(\Gamma(SL_n))$ are investigated.

We refer to [4, 5] for other terms in semigroup and graph theories, which are not explained here.

## 2. Definitions and notations

A graph $G$ consists of two pairs $(V(G), E(G))$, where the vertex set of $G$ is denoted by $V(G)$ and the edge set of $G$ is denoted by $E(G)$. If an undirected graph $G$ does not have any loops or multiple edges, then $G$ is called simple graph. In this paper, we only consider simple graphs. For any $n+1$ different vertices $u = v_1 - \cdots - v_n - v_{n+1} = v$ in $V(G)$, if there exists an edge $v_i - v_{i+1}$ in $E(G)$for each $1 \leq i \leq n$, then $u = v_1 - \cdots - v_n - v_{n+1} = v$ is called a path between $u$ and $v$, and $n$ is called the length of the path. The length of a shortest path between $u$ and $v$ in $G$ is denoted by $d_G(u,v)$. If there exist path for all two distinct vertices in $G$, then $G$ is called a connected graph.

The eccentricity of a vertex $v$ in $G$ is denoted by $ecc(G)$ and defined by

$$ecc(v) = \max\{d_G(u,v) : u \in V(G)\}.$$

Then, the diameter of $G$ is denoted by $diam(G)$ and defined by

$$diam(G) = \max\{ecc(v) : v \in V(G)\}.$$

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For $v \in V(G)$, the degree of $v$ is denoted by $deg_G(v)$ and defined as the cardinality of the adjacent vertices to $v$ in $G$. Moreover, the minimum degree of $G$ is denoted by $\delta(G)$ and defined by

$$\delta(G) = \min\{deg_G(v): v \in V(G)\}.$$

Similarly, the maximum degree of $G$ is denoted by $\Delta(G)$ and defined by

$$\Delta(G) = \max\{deg_G(v): v \in V(G)\}.$$

The length of the shortest cycle contained in a graph $G$ is called girth of $G$ and it is denoted by $gr(G)$. Moreover, if $G$ does not contain any cycles, then its girth is defined as infinity. Let $C$ be a non-empty subset of $V(G)$. If every two distinct vertices in $C$ are adjacent, then $C$ is called a clique in $G$. The number of all the vertices in any maximal clique of $G$ is called clique number of $G$, denoted by $\omega(G)$. If we color all the vertices in $G$ with the rule of no two adjacent vertices have the same color, then the minimum number of colors needed to color of $G$ is called chromatic number of $G$, denoted by $\chi(G)$. For any graph G, it is well known that $\chi(G) \geq \omega(G)$ [5].

An edge coloring of a graph is an assignment of colors to the edges of $G$ such that no two adjacent edges have the same color. The minimum required number of colors for and the edge coloring of $G$ is called the chromatic index of $G$, denoted by $\chi'(G)$.

For an undirected graph $G$, the line graph of $G$ is an undirected graph denoted by $L(G)$. The edge set of $G$ represents the vertex set of $L(G)$ and two vertices are adjacent in $L(G)$ if and only if they have common vertex in $G$. Thus, if $G$ is a connected graph, then $L(G)$ is a connected graph.

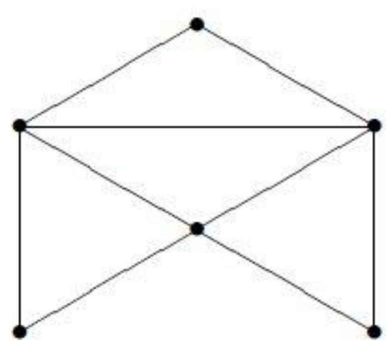
In this paper, we use $\Gamma_n$ instead of $\Gamma(SL_n)$ for convenience. Moreover, we investigate $L(\Gamma(SL_n))$ for $n \geq 3$. It is known that $\Gamma(SL_n)$ is a connected graph for $n \geq 3$, and so $L(\Gamma(SL_n))$. If $v \in V(L\Gamma_n)$, then we can consider that vertex as $(A, B)$ such that $\emptyset \neq A \subsetneq X_n$, $\emptyset \neq B \subsetneq X_n$ and $A \cup B = X_n$. In this case, we write $v \sim (A, B)$.

We find the cardinality of the vertex set of $L(\Gamma_n)$, diameter, maximum degree, minimum degree, girth, clique number, chromatic number, and degrees of all vertices in $L(\Gamma_n)$ for $n \geq 3$.

**3. Main Results**

**Example 3.1** Let $G = L(\Gamma_3)$. Then, we have $V(G) = \{v_1 \sim (\{1\}, \{2,3\}), v_2 \sim (\{2\}, \{1,3\}), v_3 \sim (\{3\}, \{1,2\}), v_4 \sim (\{1,2\}, \{2,3\}), v_5 \sim (\{1,3\}, \{2,3\}), v_6 \sim (\{1,2\}, \{1,3\})\}$ and $G$ is isomorphic to the following graph.

**Figure 1**



**Theorem 3.2.** For $n \geq 3$, $V\big(L(\Gamma_n)\big) = \frac{1}{2} . \sum_{k=1}^{n-1} \binom{n}{k} (2^k - 1)$.

**Proof.** Let $n \geq 3$. From the definiton of a line graph, it is clear that $|V\big(L(\Gamma_n)\big)| = |E(\Gamma_n)|$. If $\emptyset \neq A \subsetneq X_n$ and $|A| = k$, then $\deg_\Gamma(A) = 2^k - 1$ [6]. Moreover there are $\binom{n}{k}$ different subsets of $X_n$ with cardinality $k$. In any simple graph $G$, it is known that $|E(G)| = \frac{1}{2} |\sum_{v \in G} \deg_G(v)|$ [5]. It follows that $|V\big(L(\Gamma_n)\big)| = \frac{1}{2} \cdot \sum_{k=1}^{n-1} \binom{n}{k} (2^k - 1)$.

**Theorem 3.3**. For $n \geq 3$, $\operatorname{diam}(L(\Gamma_n)) = \begin{cases} 2 & \text{if } n = 3 \\ 3 & \text{if } n \geq 4. \end{cases}$

**Proof.** It is clear from Figure 1 that if $n = 3$, then $\operatorname{diam}(L(\Gamma_n)) = 2$. Let $n \geq 4$. Let $v_1, v_2 \in V(L(\Gamma_n))$ and $v_1 \sim (A, B)$ and $v_2 \sim (C, D)$. It follows that $A \cup B = X_n$ and $C \cup D = X_n$. There is $i \in X_n$ such that $i \notin C$.
**Case 1:** Let $i \in A$. Then, there are $v_3, v_4 \in V(L(\Gamma_n))$ such that $v_3 \sim (A, X_n \backslash \{i\})$ and $v_4 \sim (C, X_n \backslash \{i\})$. It follows that $v_1 - v_3 \in E(L(\Gamma_n))$ if $B \neq X_n \backslash \{i\}$, $v_3 - v_4 \in E(L(\Gamma_n))$ if $A \neq C$ and $v_1 - v_4 \in E(L(\Gamma_n))$ if $D \neq X_n \backslash \{i\}$. Thus, $d_{L(\Gamma_n)}(v_1, v_2) \leq 3$. Notice that, if $B = X_n \backslash \{i\}$ or $A = C$ or $D = X_n \backslash \{i\}$, then it is clear that $d_{L(\Gamma_n)}(v_1, v_2) \leq 3$.
**Case 2:** Let $i \in B$. In this case, $d_{L(\Gamma_n)}(v_1, v_2) \leq 3$ can be shown similarly.
**Case 3:** Let $i \in A \cap B$. In this case, $d_{L(\Gamma_n)}(v_1, v_2) \leq 3$ can be shown similarly.
Therefore, $\operatorname{diam}(L(\Gamma_n)) \leq 3$ for $n \geq 4$. Let $v_1 \sim (X_n \backslash \{1,2\}, X_n \backslash \{3,4\})$ and $v_2 \sim (X_n \backslash \{1,3\}, X_n \backslash \{2,4\})$. It is clear that $v_1$ and $v_2$ are not adjacent vertices in $L(\Gamma_n)$. Moreover, it is easy to see that $(N(v_1) \cup \{v_1\}) \cap (N(v_2) \cup \{v_2\}) = \emptyset$. It follows that $d_{L(\Gamma_n)}(v_1, v_2) > 2$. Thus, we conclude that $\operatorname{diam}(L(\Gamma_n)) = 3$ for $n \geq 4$.

**Theorem 3.4.** For $n \geq 3$, let $v \in V(L(\Gamma_n))$ such that $v \sim (A, B)$ and $|A| = k_1$, $|B| = k_2$. Then, $\deg_{L(\Gamma_n)}(A, B) = 2^{k_1} + 2^{k_2} - 4$.

**Proof.** For $n \geq 3$, let $v \in L(\Gamma_n)$ such that $v \sim (A, B)$ and $|A| = k_1$, $|B| = k_2$. Moreover, let $X = \{u \in V(L(\Gamma_n)) : u \sim (A, P) \in N(v) \text{and} \emptyset \neq P \subsetneq X_n\}$ and $Y = \{u \in V(L(\Gamma_n)) : u \sim (Q, B) \in N(v) \text{and} \emptyset \neq Q \subsetneq X_n\}$. It is clear that $X \cap Y = \emptyset$. Moreover, it is easy to see that $\deg_{L(\Gamma_n)}(v) = |X| + |Y|$. Since $|A| = k_1$ and $v \notin N(v)$, we have $|X| = 2^{k_1} - 2$. Similarly, we have $|Y| = 2^{k_2} - 2$. It follows that $\deg_{L(\Gamma_n)}(A, B) = 2^{k_1} + 2^{k_2} - 4$.

**Theorem 3.5**. For $n \geq 3$, $\Delta(L(\Gamma_n)) = 2^n - 4$.

**Proof.** Let $n \geq 3$ and $v \in V(L(\Gamma_n))$ such that $v \sim (A, B)$ and $|A| = k_1$, $|B| = k_2$. Then, $\deg_{L(\Gamma_n)}(A, B) = 2^{k_1} + 2^{k_2} - 4$. We have $\emptyset \neq A \subsetneq X_n$ and $\emptyset \neq B \subsetneq X_n$. Moreover $A \cup B = X_n$. So, $k_1 \leq n - 1$ and $k_2 \leq n - 1$. We may choose $k_1 = k_2 = n - 1$, it follows that $\Delta(L(\Gamma_n)) = 2^n - 4$ for $n \geq 3$.

**Theorem 3.6.** For $n \geq 3$, $\delta(L(\Gamma_n)) = \begin{cases} 2^{\frac{n-1}{2}} + 2^{\frac{n+1}{2}} - 4 & \text{if n is an odd number} \\ 2^{\frac{n+2}{2}} - 4 & \text{if n is an even number.} \end{cases}$

**Proof.** Let $n \geq 3$ and $v \in V(L(\Gamma_n))$ such that $v \sim (A, B)$ and $|A| = k_1$, $|B| = k_2$. Then, $\deg_{L(\Gamma_n)}(A, B) = 2^{k_1} + 2^{k_2} - 4$. Since $A \cup B = X_n$, we have $k_1 + k_2 \geq n$. If $n$ is an odd number, then we choose $k_1 = \frac{n-1}{2}$ and $k_2 = \frac{n+1}{2}$ for minimality. Similarly, if $n$ is an even number, we choose $k_1 = k_2 = \frac{n}{2}$ for minimality. Thus, if $n$ is an odd number, then $\delta(L(\Gamma_n)) = 2^{\frac{n-1}{2}} + 2^{\frac{n+1}{2}} - 4$ and if $n$ is an even number, then $\delta(L(\Gamma_n)) = 2^{\frac{n+2}{2}} - 4$.

**Theorem 3.7**. For $n \geq 3$, $\operatorname{gr}(L(\Gamma_n)) = 3$.

**Proof.** Let $n \geq 3$ and $v_1 \sim (X_n \backslash \{1\}, X_n \backslash \{2\})$, $v_2 \sim (X_n \backslash \{1\}, X_n \backslash \{3\})$, $v_3 \sim (X_n \backslash \{2\}, X_n \backslash \{3\})$. Then, $v_1 - v_2 - v_3 - v_1$ is a cycle in $L(\Gamma_n)$. It follows that $\operatorname{gr}(L(\Gamma_n)) = 3$ for $n \geq 3$.

**Theorem 3.8**. For $n \geq 3$, $\omega(L(\Gamma_n)) = \chi(L(\Gamma_n)) = 2^{n-1} - 1$.

**Proof.** Let $n \geq 3$. In 2016, Toker proved that $\chi'(\Gamma_n) = 2^{n-1} - 1$ [6]. Thus, from the definiton of a line graph, we have $\chi(L(\Gamma_n)) = 2^{n-1} - 1$. It is well known that for any graph $G$, $\chi(G) \geq \omega(G)$ [5]. It follows that $\omega(L(\Gamma_n) \leq 2^{n-1} - 1$. Let $W = \{(X_n \backslash \{1\}, \{1\} \cup B) : \emptyset \subseteq B \subsetneq X_n \backslash \{1\}\}$. It is clear that $|W| = 2^{n-1} - 1$, moreover $W$ is a clique in $L(\Gamma_n)$. It follows that $\omega(L(\Gamma_n)) = 2^{n-1} - 1$. Thus, $\omega(L(\Gamma_n)) = \chi(L(\Gamma_n)) = 2^{n-1} - 1$ for $n \geq 3$.

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