

RESEARCH ARTICLE**Some Topological Indices on a Graph of Finite Free Semilattices****Kemal Toker***Assoc. Prof. Dr., Department of Mathematics, Harran University, Şanlıurfa, Turkey***Corresponding Author:** Kemal Toker **E-mail:** ktoker@harran.edu.tr**ABSTRACT**

Let SL_n be the finite free semilattice on $X_n = \{1, 2, \dots, n\}$ and Γ_n be the zero-divisor graph of SL_n . In this paper, we find the first Zagreb index, second Zagreb index, and Sombor index of Γ_n for $n \geq 3$.

KEYWORDS

Finite free semilattice, Zagreb index, Sombor index.

ARTICLE INFORMATION**ACCEPTED:** 20 March 2026**PUBLISHED:** 01 April 2026**DOI:** 10.32996/jmss.2026.7.4.1**1. Introduction**

Let G be a graph. A graph G consists of two pairs $(V(G), E(G))$, where the vertex set of G is denoted by $V(G)$, and the edge set of G is denoted by $E(G)$. For any $n + 1$ different vertices $u = v_1 - \dots - v_n - v_{n+1} = v$ in $V(G)$, if there exists an edge $v_i - v_{i+1}$ in $E(G)$ for each $1 \leq i \leq n$, then $u = v_1 - \dots - v_n - v_{n+1} = v$ is called a path between u and v . A graph G is called a connected graph if there exists a path from u to v for every pair of distinct vertices $u, v \in V(G)$. If G does not have any loops or multiple edges, then G is called a simple graph. In this paper, we only consider simple graphs. The eccentricity of a vertex v in G is denoted by $ecc(G)$ and defined by $ecc(v) = \max \{d_G(u, v) : u \in V(G)\}$. The diameter of G is denoted by $diam(G)$ and defined by $diam(G) = \max \{ecc(v) : v \in V(G)\}$. The length of the shortest cycle contained in a graph G is called the girth of G and it is denoted by $gr(G)$. Moreover, if G does not contain any cycles, then its girth is defined as infinity. The degree of a vertex $v \in V(G)$ is the number of vertices adjacent to v , denoted by $deg_G(v)$. Among all degrees, the maximum degree is denoted by $\Delta(G)$, and the minimum degree is denoted by $\delta(G)$. Let C be a non-empty subset of $V(G)$. If every two distinct vertices in C are adjacent, then C is called a clique in G . The number of all the vertices in any maximal clique of G is called clique number of G . The clique number of G is denoted by $\omega(G)$. If we color all the vertices in G with the rule that no two adjacent vertices have the same color, then the minimum number of colors needed to color of G is called chromatic number of G . The chromatic number of G is denoted by $\chi(G)$. For $V' \subseteq V(G)$, the induced subgraph $G' = (V', E')$ is a subgraph of G such that E' consists of those edges whose endpoints are in V' . If $\omega(H) = \chi(H)$ for every induced subgraph H of G , then G is called a perfect graph; otherwise, it is called imperfect graph. Let D be a nonempty subset of the vertex set $V(G)$ of G . If, for each $u \in V(G) \setminus D$, there exists $v_u \in D$ such that $u - v_u \in E(G)$, then D is called a dominating set. The cardinality of minimum cardinality of a dominating set of G is called dominating number of G . An independent set of a graph G is a subset of vertices $V(G)$ such that no two vertices in the subset represent an edge of G . The cardinality of maximum cardinality of an independent set of G is called independence number of G . An edge coloring of a graph is an assignment of colors to the edges of G such that no two adjacent edges have the same color. The minimum required number of colors for and the edge coloring of G is called the chromatic index of G . For $|V(G)| = n$, the adjacency matrix of G is a binary matrix of order n . The entries of the adjacency matrix are 1 if two vertices are neighbors of each other and 0 otherwise.

The zero-divisor graph of a commutative ring is defined by Beck; however, the zero element of the ring is a vertex in Beck's definition [2]. Subsequently, Anderson and Livingston redefined the zero-divisor graph of a commutative ring which is the standard zero-divisor graph of a commutative ring [1]. Let R be a commutative ring and $Z(R)$ be the set of zero-divisors of R .

The zero-divisor graph of R is denoted by $\Gamma(R)$. It is an undirected graph with vertex set $Z(R)^* = Z(R) \setminus \{0\}$ and distinct vertices x and y are adjacent in $\Gamma(R)$ if and only if $xy = 0$. Anderson and Livingston proved that $\Gamma(R)$ is a connected graph and $diam(\Gamma(R)) \leq 3$ [1]. Anderson and Livingston proved that if $\Gamma(R)$ contains a cycle, then $gr(\Gamma(R)) \leq 7$ [1]. Moreover, Anderson and Livingston showed that if R is a commutative Artinian ring and $\Gamma(R)$ contains a cycle, then $gr(\Gamma(R)) \leq 4$ [1].

Let S be a commutative semigroup with zero and $Z(S)$ be the set of zero-divisors of S . Demeyer et al. defined the zero-divisor graph of S [3]. The zero-divisor graph of S is denoted by $\Gamma(S)$ and it is an undirected graph with vertex set $Z(S)^* = Z(S) \setminus \{0\}$ and distinct two vertices x and y are adjacent in $\Gamma(S)$ if and only if $xy = 0$. Demeyer et al. proved that $\Gamma(S)$ is a connected graph and $diam(\Gamma(S)) \leq 3$, and if $\Gamma(S)$ does not contain a cycle, then $\Gamma(S)$ is a connected subgraph of two star graphs whose centers are connected by a single edge. Moreover, they showed that the number of minimal ideals of S gives a lower bound for the clique number of $\Gamma(S)$ [3].

Let $n \in \mathbb{Z}^+$ and $X_n = \{1, 2, \dots, n\}$. Let SL_n be the set consisting of all subsets of X_n , except the empty set. SL_n is a commutative semigroup of idempotents with the multiplication $A \cdot B = A \cup B$ for $A, B \in SL_n$, and it is called the free semilattice on X_n . The zero-divisor graph of SL_n is denoted by $\Gamma(SL_n)$, and Toker investigated many properties of $\Gamma(SL_n)$ for $n \geq 3$ in 2016 [10]. For $n \geq 3$, Toker proved that $\Gamma(SL_n)$ is a connected graph and $diam(\Gamma(SL_n)) = 3$. Moreover, the degree of any vertex, domination number, independence number, clique number, chromatic number, and chromatic index of $\Gamma(SL_n)$ have been determined [10]. In this paper, we use Γ_n instead of $\Gamma(SL_n)$ for convenience.

In 1972, Gutman and Trinajestic defined the first Zagreb index of a graph [6]. Let G be a graph. The first Zagreb index of G is denoted by $M_1(G)$, where

$$M_1(G) = \sum_{v \in V(G)} (deg_G(v))^2.$$

The second Zagreb index of G was defined in 1975 [7], second Zagreb index of G is denoted by $M_2(G)$, where

$$M_2(G) = \sum_{u-v \in E(G)} deg_G(u) \cdot deg_G(v).$$

Moreover, the first Zagreb index of G is equal $\sum_{u-v \in E(G)} [deg_G(u) + deg_G(v)]$, which is an alternative equation [4].

In 2021, Gutman introduced the Sombor index of a graph [5]. For a graph G , the sombor index of G is denoted by $SO(G)$ and

$$SO(G) = \sum_{u-v \in E(G)} \sqrt{(deg_G(u))^2 + (deg_G(v))^2}.$$

Moreover, the sombor index is one of the most recently introduced topological indices in chemical graph theory.

We refer to [8, 9] for other terms in semigroup and graph theories, which are not explained here. In this paper, we calculate the first Zagreb index, second Zagreb index, and Sombor index of Γ_n for $n \geq 3$. In addition, we provide an example.

2. Main Results

Let G be graph and $v \in V(G)$. Moreover, let

$$x(v) = \sum_{u-v \in E(G)} deg_G(u) \cdot deg_G(v)$$

$$y(v) = \sum_{u-v \in E(G)} \sqrt{(deg_G(u))^2 + (deg_G(v))^2}.$$

Thus, it is easy to see that

$$M_2(G) = \sum_{u-v \in E(G)} deg_G(u) \cdot deg_G(v) = \frac{1}{2} \sum_{v \in V(G)} x(v),$$

$$SO(G) = \sum_{u-v \in E(G)} \sqrt{(deg_G(u))^2 + (deg_G(v))^2} = \frac{1}{2} \sum_{v \in V(G)} y(v).$$

For $n \geq 3$, let $A \in V(\Gamma_n)$ and $|A| = k$ where $1 \leq k \leq n - 1$. Then, $deg_{\Gamma_n}(v) = 2^k - 1$, and there are $\binom{n}{k}$ vertices in Γ_n whose vertex degrees are $2^k - 1$ [10]. In the following theorem, we give the first Zagreb index of Γ_n for $n \geq 3$.

Theorem 2.1. For $n \geq 3$, we have $M_1(\Gamma_n) = \sum_{k=1}^{n-1} \binom{n}{k} (2^k - 1)^2$.

Proof. For $n \geq 3$, let $A \in V(\Gamma_n)$ and $|A| = k$ for $1 \leq k \leq n - 1$. Since $deg_{\Gamma_n}(v) = 2^k - 1$ and there are $\binom{n}{k}$ vertices in Γ_n whose vertex degrees are $2^k - 1$ and $M_1(\Gamma_n) = \sum_{v \in V(\Gamma_n)} (deg_G(v))^2$, we have $M_1(\Gamma_n) = \sum_{k=1}^{n-1} \binom{n}{k} (2^k - 1)^2$.

In the following theorem, we give the second Zagreb index of Γ_n for $n \geq 3$.

Theorem 2.2. For $n \geq 3$, we have $M_2(\Gamma_n) = \frac{1}{2} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \binom{n}{k} (2^k - 1) (2^{n-k+r} - 1) \binom{k}{r}$.

Proof. Let $n \geq 3$ and $A \in V(\Gamma_n)$ where $|A| = k$ for $1 \leq k \leq n - 1$. Moreover, let

$$W_r = \{ \bar{A} \cup B : A - \bar{A} \cup B \in E(\Gamma_n), \emptyset = B \subsetneq A \text{ and } |B| = r \}$$

for $0 \leq r \leq k - 1$. Let $N(A)$ be the set of all adjacent vertices of A in Γ_n . It is easy to see that $W_a \cap W_b = \emptyset$ where $a \neq b$ and $N(A) = \cup_{p=0}^{k-1} W_p$. It is clear that $|W_r| = \binom{k}{r}$, and if $Q \in W_r$, then $deg_{(\Gamma_n)}(Q) = 2^{n-k+r} - 1$. It follows that

$$x(A) = \sum_{r=0}^{k-1} (2^k - 1) (2^{n-k+r} - 1) \binom{k}{r}.$$

Since there are $\binom{n}{k}$ vertices in Γ_n whose vertex degrees are $2^k - 1$, we conclude that

$$M_2(\Gamma_n) = \frac{1}{2} \sum_{A \in V(\Gamma_n)} x(A) = \frac{1}{2} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \binom{n}{k} (2^k - 1) (2^{n-k+r} - 1) \binom{k}{r}.$$

In the following theorem, we give the Sombor index of Γ_n for $n \geq 3$.

Theorem 2.3. For $n \geq 3$, we have $SO(\Gamma_n) = \frac{1}{2} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \binom{n}{k} \binom{k}{r} \sqrt{(2^k - 1)^2 + (2^{n-k+r} - 1)^2}$.

Proof. Let $n \geq 3$ and $A \in V(\Gamma_n)$ where $|A| = k$ for $1 \leq k \leq n - 1$. Moreover, let

$$W_r = \{ \bar{A} \cup B : A - \bar{A} \cup B \in E(\Gamma_n), \emptyset = B \subsetneq A \text{ and } |B| = r \}$$

for $0 \leq r \leq k - 1$. Let $N(A)$ be the set of all adjacent vertices of A in Γ_n . It is easy to see that $W_a \cap W_b = \emptyset$ where $a \neq b$ and $N(A) = \cup_{p=0}^{k-1} W_p$. It is clear that $|W_r| = \binom{k}{r}$, and if $Q \in W_r$, then $deg_{(\Gamma_n)}(Q) = 2^{n-k+r} - 1$. It follows that $y(A) = \sum_{r=0}^{k-1} \sqrt{(2^k - 1)^2 + (2^{n-k+r} - 1)^2} \binom{k}{r}$. Since there are $\binom{n}{k}$ vertices in Γ_n whose vertex degrees are $2^k - 1$, we conclude that

$$SO(\Gamma_n) = \frac{1}{2} \sum_{A \in V(\Gamma_n)} y(A) = \frac{1}{2} \sum_{k=1}^{n-1} \sum_{r=0}^{k-1} \binom{n}{k} \binom{k}{r} \sqrt{(2^k - 1)^2 + (2^{n-k+r} - 1)^2}.$$

Example 2.4. Let $\Gamma = \Gamma_4$ and A be the adjacency matrix of Γ . Then, we have

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Moreover, Γ is isomomorphic to the following graph in Figure 1.

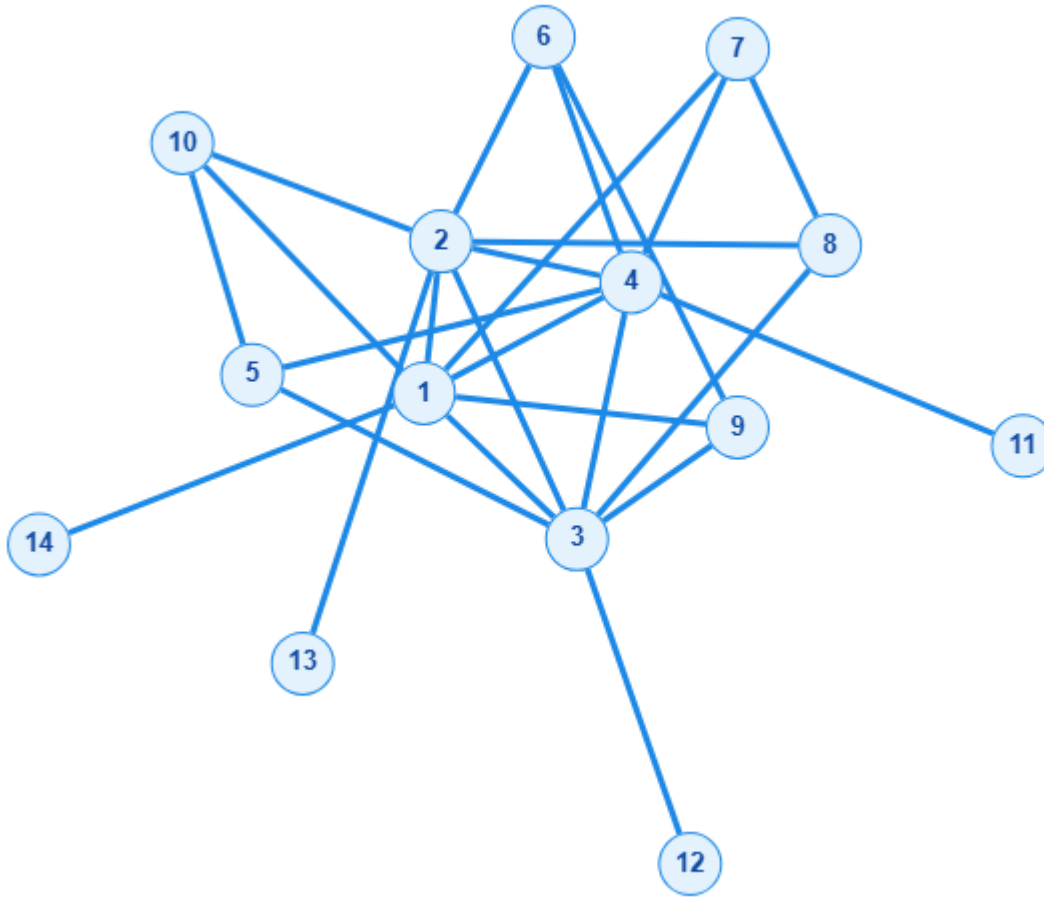


Figure 1

In addition,

$$M_1(\Gamma) = \sum_{k=1}^3 \binom{4}{k} (2^k - 1)^2 = 236$$

$$M_2(\Gamma) = \frac{1}{2} \sum_{k=1}^3 \sum_{r=0}^{k-1} \binom{4}{k} (2^k - 1) (2^{4-k+r} - 1) \binom{k}{r} = 642$$

$$SO(\Gamma) = \frac{1}{2} \sum_{k=1}^3 \sum_{r=0}^{k-1} \binom{4}{k} \binom{k}{r} \sqrt{(2^k - 1)^2 + (2^{4-k+r} - 1)^2} = 71\sqrt{2} + 21\sqrt{58}.$$

3. Conclusions

Studying graphs with respect to topological indices in chemical graph theory is important. Many topological indices have been introduced, and several of them have been found to have various applications. The Sombor index has received considerable attention in mathematics and chemistry. In this paper, we study the first Zagreb index, second Zagreb index, and Sombor index of zero-divisor graphs of finite free semilattices.

Funding: This research received no external funding.

Conflicts of Interest: Declare conflicts of interest or state "The authors declare no conflict of interest."

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References

- [1] Anderson, D. F. & Livingston, P. S. (1999). The zero-divisor graph of a commutative ring, *J. Algebra*, 217 (2), 434-447. <https://doi.org/10.1006/jabr.1998.7840>
- [2] Beck, I. (1988). Coloring of commutative rings. *J. Algebra*, 116 (1), 208-226. [https://doi.org/10.1016/0021-8693\(88\)90202-5](https://doi.org/10.1016/0021-8693(88)90202-5)
- [3] DeMeyer, F. & McKenzie, T. & Schneider, K. (2002). The zero-divisor graph of a commutative semigroup. *Semigroup Forum*, 65, 206-214. <https://doi.org/10.1007/s002330010128>
- [4] Došlić, T., Furtula, B., Graovac, A., Gutman, I., Moradi, S., & Yarahmadi, Z. (2011). On Vertex–Degree–Based Molecular Structure Descriptors. *Match-Communications in Mathematical and in Computer Chemistry*, 66, 613–626.
- [5] Gutman, I. (2021). Geometric approach to degree-based topological indices: Sombor indices. *MATCH Commun. Math. Comput. Chem.* 86, 11-16.
- [6] Gutman, I. and Trinajstić, N. (1972). Graph Theory and Molecular Orbitals, Total π -Electron Energy of Alternant Hydrocarbons. *Chemical Physics Letters*, 17, 535-538. [https://doi.org/10.1016/0009-2614\(72\)85099-1](https://doi.org/10.1016/0009-2614(72)85099-1)
- [7] Gutman, I., Ruscic, B., Trinajstić, N. and Wilcox, C.F. (1975) Graph Theory and Molecular Orbitals. XII. Acyclic Ployenes. *The Journal of Chemical Physics*, 62, 3399-3405. <https://doi.org/10.1063/1.430994>
- [8] Howie, J. M. (1995). *Fundamentals of Semigroup Theory*. New York, NY, USA: Oxford University Press.
- [9] Thulasiraman, K. & Arumugam, S. & Brandstädt, A. & Nishizeki, T. (2015). *Handbook of Graph Theory, Combinatorial Optimization, and Algorithms*. CRC Press: Boca Raton, FL, USA.
- [10] Toker, K. (2016). On the zero-divisor graphs of finite free semilattices. *Turkish Journal of Mathematics*, 40 (4): 824-831. <https://doi.org/10.3906/mat-1508-38>