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| RESEARCH ARTICLE

A Study on Saccheri Gyroquadriterals in Einstein Gyrovector Spaces

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| ABSTRACT

In this paper, we investigate the relationship between the Lorentz factors of the gyrodistances from a given point P inside a Saccheri gyroquadrilateral ***ABCD*** to its vertices within an Einstein gyrovector space.

| KEYWORDS

Saccheri Gyroquadrilaterals, Einsteinian-Phythagorean Identities, Gamma factors

| ARTICLE INFORMATION

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1. Introduction

In Euclidean geometry, the British Flag theorem states that if a point ***P*** is chosen inside a rectangle ***ABCD*** then the sums of squares of the Euclidean distance from ***P*** to the two opposite corners of the rectangle equals the sum to the other two opposite corners, that is as an equation[2]

$$AP^2 + CP^2 = BP^2 + DP^2.$$

This theorem is an application of the Pythagorean theorem in plane geometry. Hence the theorem can also be thought as a generalization of the Pythagorean teorem.

Hyperbolic geometry, is known as a type of non-Euclidean geometry that appeared in the first half of the 19th century. Although Euclidean geometry and hyperbolic geometry have common concepts as distance, angle, both these geometries have many different. Hyperbolic Geometry has many models such as: Poincare' disc model, Einstein relativistic velocity model, etc.

Einstein gyrovector spaces form the algebraic setting for the Beltrami-Klein ball model of Hyperbolic Geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean geometry.

Many objects of Euclidean geometry (triangles, squares, paralelograms, conics, etc.) have been studied by various researchers, and the hyperbolic geometric properties of these objects (gyrotriangles, gyrosquares, gyroparalelograms, gyroconics, etc.) have been presented[1,3,4].

In 1733, Milan, Giovanni Giralomo Saccheri wrote *Euclides Vindicatus.* In the book Saccheri invented what is now called hyperbolic geometry. The Saccheri hyperbolic quadrilateral, called in gyrolanguage the Saccheri guroquadrilateral that has two opposing sides that are congruent and perpendicular to a third side in a gyrovector space as shown in Fig. for an Einstein gyrovector space. For more details see [5,9].

In this paper, we present an equality involving gamma factors of the gyrodistances from a given point $\boldsymbol{P}$ to the vertices of a***n*** ***ABCD*** Sacchari gyroquadrilateral in an Einstein gyrovector space.

2. Preliminaries

In this paper, we study in an Einstein gyrovector space who was introduced by A. A. Ungar[see 6,7,8].

Definition 2.1. (Gyrogroup) A groupoid $(\mathbb{G},\oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In $\mathbb{G}$ there is at least one element, **0,** called left identity, satisfying

$$\mathbf{0}\oplus\boldsymbol{a}=\boldsymbol{a}$$

for all $\boldsymbol{a}\in\mathbb{G}$. There is an element $\mathbf{0}\in\mathbb{G}$ for each $\boldsymbol{a}\in\mathbb{G}$ there is an element $\ominus\boldsymbol{a}\in\mathbb{G}$, called a left inverse of $\boldsymbol{a}$**,** satisfying

$$\ominus\boldsymbol{a}\oplus\boldsymbol{a}=\mathbf{0}\ .$$

Moreover, for any $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in\mathbb{G}$ there exit a unique element $gyr[\boldsymbol{a},\boldsymbol{b}]\boldsymbol{c}\in\mathbb{G}$ such that binary operation obeys the left gyroassociative law

$$\boldsymbol{a}\oplus(\boldsymbol{b}\oplus\boldsymbol{c})=(\boldsymbol{a}\oplus\boldsymbol{b})\oplus gyr[\boldsymbol{a},\boldsymbol{b}]\boldsymbol{c}.$$

The map $gyr:\mathbb{G}\to\mathbb{G}$ is given by $\boldsymbol{c}\mapsto gyr[\boldsymbol{a},\boldsymbol{b}]\boldsymbol{c}$ is an automorphism of the groupoid $(\mathbb{G},\oplus)$, that is,

$$gyr[\boldsymbol{a},\boldsymbol{b}]\in \boldsymbol{Aut}(\mathbb{G},\oplus)$$

and the automorphism $gyr[\boldsymbol{a},\boldsymbol{b}]$ of automorphism of $\mathbb{G}$ is called the gyroautomorphism of $\mathbb{G}$ generated by $\boldsymbol{a},\boldsymbol{b}\in\mathbb{G}$. Finally, the gyroautomorphism of $\mathbb{G}$ generated by $\boldsymbol{a},\boldsymbol{b}\in\mathbb{G}$ possesses the left loop property

$$gyr[\boldsymbol{a},\boldsymbol{b}]=gyr[\boldsymbol{a}\oplus b,\boldsymbol{b}]\textbf{.}$$

Additionally, if the binary operation " $\oplus$ " obeys the gyrocommutative law

$$\boldsymbol{a}\oplus\boldsymbol{b}=gyr[\boldsymbol{a},\boldsymbol{b}](\boldsymbol{b}\oplus\boldsymbol{a})$$

for all $\boldsymbol{a},\boldsymbol{b}\in\mathbb{G}$, then $(\mathbb{G},\oplus)$ is called a gyrocommutative gyrogroup.

Definition 2.2. (Einstein addition $\oplus$) Let $\mathbb{V}$ be a real inner product space and let $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$,

$$\mathbb{V}_s=\{\boldsymbol{v}\in\mathbb{V}:\|\boldsymbol{v}\|<s\},$$

where $s>0$ is an arbitrary fixed constant. Einstein addition $\oplus$ is a binary operation in $\mathbb{V}_s$ given by the equation

$$\boldsymbol{u}\oplus\boldsymbol{v}=\frac{1}{1+\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}}\{\boldsymbol{u}+\frac{1}{\gamma_{\boldsymbol{u}}}\boldsymbol{v}+\frac{1}{s^2}\frac{\gamma_{\boldsymbol{u}}}{1+\gamma_{\boldsymbol{u}}}(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{v}\}$$

where $\gamma_{\boldsymbol{u}}$ is the gamma factor

$$\gamma_{\boldsymbol{u}}=\frac{1}{\sqrt{1-\frac{\|\boldsymbol{u}\|^2}{s^2}}}\geq 1$$

in the s-ball $\mathbb{V}_s$, and where $\cdot$ and $\|.\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$.

In the Newtonian limit, $s\to\infty$, s-ball $\mathbb{R}_s^n$ expands to the whole of its space $\mathbb{R}^n$, and Einstein addition $\oplus$ in $\mathbb{R}_s^n$ reduces to vector addition $+$ in $\mathbb{R}^n$.

Theorem 2.3. $(\mathbb{R}_s^n,\oplus)$ Einstein groupoid is a gyrocommutative gyrogroup.

Some gyrocommutative gyrogroups admit scalar multiplication, giving rise to gyrovector spaces.

Definition 2.4. (Gyrovector spaces)A $(\mathbb{G},\oplus,\otimes)$ gyrovector space is a gyrocommutative gyrogroup $(\mathbb{G},\oplus)$ that obeys the following axioms:

1. $gyr[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{a}\cdot gyr[\boldsymbol{u},\boldsymbol{v}]\boldsymbol{b}=\boldsymbol{a}\cdot\boldsymbol{b}$ for all points $\boldsymbol{a},\boldsymbol{b},\boldsymbol{u},\boldsymbol{v}\in\mathbb{G}$.

2. $\mathbb{G}$ admits a scalar multiplication, $\otimes$, possessing following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points and $\boldsymbol{a} \in \mathbb{G}$:
   - $1\otimes\boldsymbol{a} = \boldsymbol{a}$
   - $(r_1 + r_2)\otimes\boldsymbol{a} = (r_1\otimes \boldsymbol{a}) \oplus (r_2\otimes\boldsymbol{a})$
   - $(r_1 r_2)\otimes\boldsymbol{a} = r_1\otimes(r_2\otimes\boldsymbol{a})$
   - $\frac{|r|\otimes\boldsymbol{a}}{\|r\otimes\boldsymbol{a}\|} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \quad \boldsymbol{a} \neq \boldsymbol{0} \ , \ r \neq 0$
   - $\mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}](r\otimes\boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u},\boldsymbol{v}](\boldsymbol{a})$
   - $\mathrm{gyr}[r_1\otimes \boldsymbol{v}, r_2\otimes \boldsymbol{v}] = I$
3. Real vector space structure ($\|\mathbb{G}\|,\oplus,\otimes$) for the set $\|\mathbb{G}\|$ of one-dimensional "vectors"
$$\|\mathbb{G}\| \coloneqq \{\mp\|\boldsymbol{a}\|:\ \boldsymbol{a} \in \mathbb{G}\} \subset \mathbb{R}$$
   with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{G}$,
   - $\|r\otimes\boldsymbol{a}\| = |r|\otimes\|\boldsymbol{a}\|$
   - $\|\boldsymbol{a}\oplus\boldsymbol{b}\| \leq \|\boldsymbol{a}\|\oplus\|\boldsymbol{b}\|$

Theorem 2.5. An Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is an Einstein gyrocommutative gyrogroup $(\mathbb{R}_s^n, \oplus)$ with scalar multiplication $\otimes$ given by

$$r\otimes\boldsymbol{v} = s\frac{\left(1+\frac{\|\boldsymbol{v}\|}{s}\right)^r - \left(1-\frac{\|\boldsymbol{v}\|}{s}\right)^r}{\left(1+\frac{\|\boldsymbol{v}\|}{s}\right)^r + \left(1-\frac{\|\boldsymbol{v}\|}{s}\right)^r}\frac{v}{\|\boldsymbol{v}\|} = stanh(rtanh^{-1}\frac{\|\boldsymbol{v}\|}{s})\frac{v}{\|\boldsymbol{v}\|}$$

Definition 2.6. (Gyrodistance) Let $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ be an Einstein gyrovector space. Its gyrometric is given by the gyrodistance function $d_\oplus\colon \mathbb{R}_s^n \times \mathbb{R}_s^n \longrightarrow \mathbb{R}^{\geq 0} \coloneqq \{r \in \mathbb{R}\colon r \geq 0\}$,

$$d_\oplus(\boldsymbol{a},\boldsymbol{b}) = \|\ominus \boldsymbol{a} \oplus \boldsymbol{b}\| = \|\boldsymbol{b} \ominus \boldsymbol{a}\|$$

where $d_\oplus(\boldsymbol{a},\boldsymbol{b})$ is the gyrodistance of $\boldsymbol{a}$ and $\boldsymbol{b}$**.**

Definition 2.7. (Gyroline-Gyroline segment) The unique Einstein gyroline L_{AB} that passes two given points A and B in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is represented by the equation

$$L_{AB} = A \oplus (\ominus A \oplus B)\otimes t$$

$t \in \mathbb{R}$.

Definition 2.8. (Gyroangle) Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$ be three distinct points and $\ominus \boldsymbol{A} \oplus \boldsymbol{B}$, $\ominus \boldsymbol{A} \oplus \boldsymbol{C}$ be two rooted gyrovectors with a common tail $\boldsymbol{A}$. They include the gyroangle $\boldsymbol{\alpha} = \angle\boldsymbol{BAC} = \angle\boldsymbol{CAB}$, the radian measure of which is given by the equation
$$\boldsymbol{cos\alpha} = \frac{\ominus\boldsymbol{A}\oplus\boldsymbol{B}}{\|\ominus\boldsymbol{A}\oplus\boldsymbol{B}\|}\cdot\frac{\ominus\boldsymbol{A}\oplus\boldsymbol{C}}{\|\ominus\boldsymbol{A}\oplus\boldsymbol{C}\|}.$$

Definition 2.9.(Gyrotriangle). A gyrotriangle $\boldsymbol{ABC}$ in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is a object formed by the three points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$, called the vertices of the triangle, and the gyrovectors $\ominus \boldsymbol{A} \oplus \boldsymbol{B}$, $\ominus \boldsymbol{B} \oplus \boldsymbol{C}$ and $\ominus \boldsymbol{C} \oplus \boldsymbol{A}$**,** called the sides of the triangle. These are respectively, the sides opposite to the vertices **C,A** and **B.** The gyrotriangle sides generate the three gyrotriangle gyroangles $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ at the respective vertices **A, B** and **C**.

Theorem 2.10. (Law of Gyrocosine) Let $\boldsymbol{ABC}$ be a gyrotriangle in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, with vertices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$,and sides $\boldsymbol{c} = \ominus \boldsymbol{A} \oplus \boldsymbol{B}, \boldsymbol{a} = \ominus \boldsymbol{B} \oplus \boldsymbol{C}$ and $\boldsymbol{b} = \ominus \boldsymbol{C} \oplus \boldsymbol{A}$**,** with gyroangles $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ at the vertices **A, B** and **C**. Then we have the law of gyrocosines
$$\gamma_c = \gamma_a\gamma_b(1 - b_s c_s cos\gamma)$$
where $a = \|\boldsymbol{a}\|, b = \|\boldsymbol{b}\|, c = \|\boldsymbol{c}\|$ and $b_s = b/s$, etc.

Definition 2.11. (Right Gyroangle) A right gyroangle γ is a gyroangle measuring $\frac{\pi}{2}$ radians.

Theorem 2.12. A gyrotriangle $\boldsymbol{ABC}$ in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is a right gyrotriangle with gyrolegs $\boldsymbol{a}, \boldsymbol{b}$ and gyrohypotenuse $\boldsymbol{c}$, if and only if
$$\gamma_c = \gamma_a\gamma_b.$$

Theorem 2.13. (Side-Side-Side (SSS) Gyrotriangle Congruence Theorem) If, in two triangles, three sides of one are congruent to three sides of the other, then the two gyrotriangles are congruent.

Theorem 2.14. (Side-gyroAngle-Side (SAS) Gyrotriangle Congruence Theorem) If, in two triangles, two sides and the included gyroangle of one, are congruent to two sides and the included gyroangle of the other, then the two gyrotriangles are congruent.

3. Saccheri Gyroquadrilaterals

Definition 3.1. Saccheri gyroquadriteral is a gyroquadriteral that has two opposing sides that are congruent and perpendicular to a third side in a gyrovector space, as shown Fig. 3.1. for an Einstein gyrovector space. The congruent gyrosegments ***AD*** and ***BC*** are the sides, or legs, of Saccheri gyroquadriteral. The gyrosegment ***AB*** is the base and the gyrosegment ***CD*** is the summit of the Saccheri gyroquadrilateral. The gyroangles $\angle \boldsymbol{ADC}$ and $\angle \boldsymbol{BCD}$**,** called the summit gyroangles of the Saccheri gyroquadrilateral.

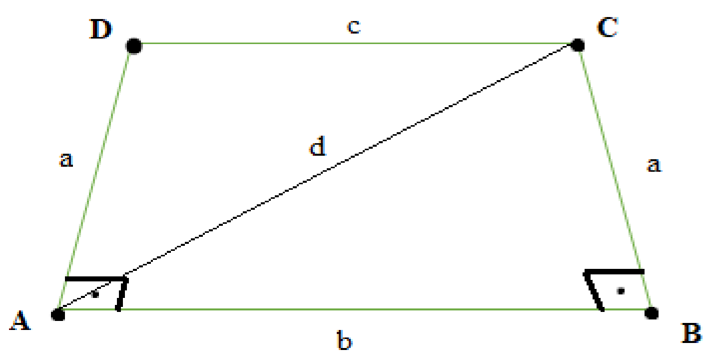

Fig. 3.1. The Saccheri Gyroquadrilateral in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$

The two gyrodiagonals, ***AC*** and ***BD***, of the gyroquadrilateral ***ABCD*** on the base ***AB*** in Fig. 3.1. have equal gyrolenghts of the gyroquadrilateral ***ABCD*** have equal measures by SSS congruency, Theorem 8.31. Let α be the measure of each of the two summit gyroangles of the Saccheri gyroquadrilareal ***ABCD*** on the base ***AB*** in Fig. 3.1,

$$\alpha = \angle \boldsymbol{ADC} = \angle \boldsymbol{BCD}.$$

Furthermore, as shown in Fig. 3.1., a is the gyrolength of each of the congruent sides, ***AD*** and ***BC*** of the Saccheri gyroquadriteral ***ABCD***, b is the gyrolength of its base, c is the gyrolength of its summit, and d is the gyrolength of each of its two gyrodiagonals. In symbols,

$$a = \|\ominus \boldsymbol{B} \oplus \boldsymbol{C}\| = \|\ominus \boldsymbol{A} \oplus \boldsymbol{D}\|$$

$$d = \|\ominus \boldsymbol{A} \oplus \boldsymbol{C}\| = \|\ominus \boldsymbol{B} \oplus \boldsymbol{D}\|$$

$$b = \|\ominus \boldsymbol{A} \oplus \boldsymbol{B}\|$$

$$c = \|\ominus \boldsymbol{D} \oplus \boldsymbol{C}\|$$

Theorem 3.5. Let $\boldsymbol{ABCD}$ be a Saccheri gyroquadrilateral on the base $\boldsymbol{AB}$ in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, with base gyrolength *b* and sides gyrolength a in Fig. . Then, each of its gyrodiagonals has gyrolength *d,* its summit has gyrolength c, and each of its summit gyroangles has measure α, where

$$d_s = \frac{\sqrt{\gamma_a^2 \gamma_b^2 - 1}}{\gamma_a \gamma_b},$$

$$c_s = \frac{\sqrt{\gamma_b - 1}\sqrt{2 + \gamma_a^2(\gamma_b - 1)}}{1 + \gamma_a^2(\gamma_b - 1)} \gamma_a$$

and

$$cos\alpha = \frac{\sqrt{\gamma_a^2 - 1}\sqrt{\gamma_b - 1}}{\sqrt{2 + \gamma_a^2(\gamma_b - 1)}}.$$

Furthermore,

$$\gamma_a^2 = \frac{\gamma_c - 1}{\gamma_b - 1}$$

and

$$c > b.$$

4. Main Result

As an application of Einteinian-Pythagorian identities and in Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, we verify the following theorem:

Theorem 4.1. Let $\boldsymbol{ABCD}$ be a Saccheri gyroquadrilateral on the base $\boldsymbol{AB}$ in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, with base gyrolength *b*, sides gyrolength *a*, base gyrolength *b*, gyrodiagonal gyrolength *d*, summit gyrolength *c* in Fig. 4.1., and the $\boldsymbol{P}$ be a point inside the $\boldsymbol{ABCD}$ Saccheri gyroquadrilateral. Then,

$$\gamma_t\gamma_y - \gamma_x\gamma_z = (1 + \gamma_x\gamma_y)(\gamma_y - \gamma_x) \qquad 4.1)$$

and

$$\gamma_z(\gamma_t - 1 + \gamma_x^2)^2 = \gamma_x^3(\gamma_t - 1 + \gamma_z^2 - \gamma_z) \qquad (4.2)$$

where $\|\ominus \boldsymbol{A} \oplus \boldsymbol{P}\| = \boldsymbol{x}$, $\|\ominus \boldsymbol{B} \oplus \boldsymbol{P}\| = \boldsymbol{y}$, $\|\ominus \boldsymbol{C} \oplus \boldsymbol{P}\| = \boldsymbol{z}$, $\|\ominus \boldsymbol{D} \oplus \boldsymbol{P}\| = \boldsymbol{t}$. .

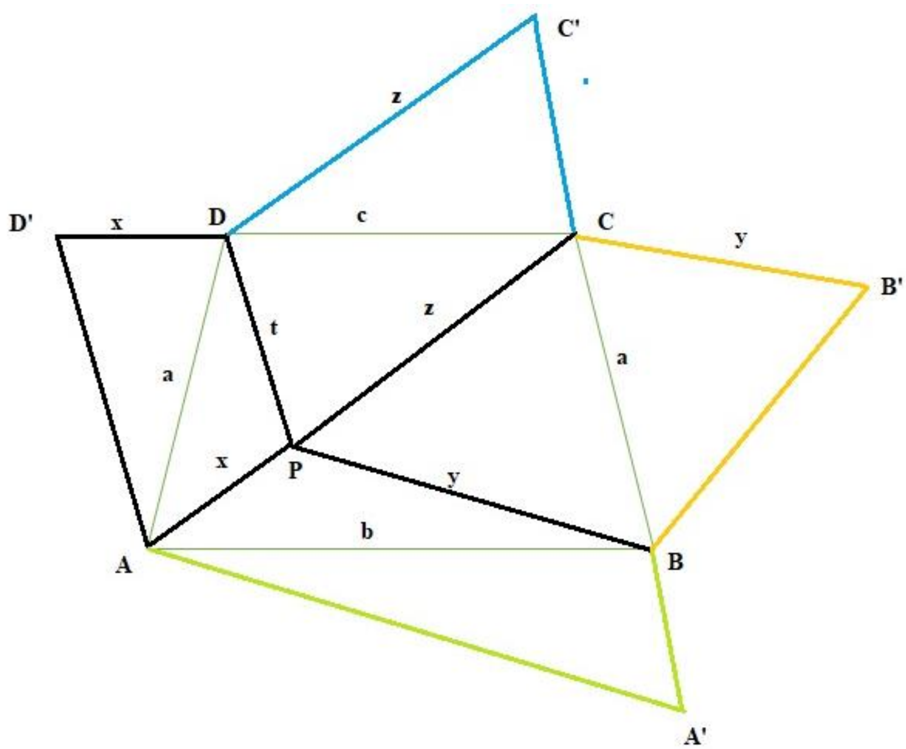

Fig. 4.1. The Saccheri Gyroquadrilateral

Proof : Let $\boldsymbol{ABCD}$ be a Saccheri gyroquadrilateral on the base $\boldsymbol{AB}$ in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, with base gyrolength *b* and sides gyrolength a, and $\boldsymbol{P}$ be a point inside the Saccheri gyroquadrilateral $\boldsymbol{ABCD}$**.** Let us connect point $\boldsymbol{P}$ to the vertices of Saccheri gyroquadriteral $\boldsymbol{ABCD}$ with gyroline segments,and gyrolength of these gyrolines x,y,z and t. In symbols

$$\|\ominus \boldsymbol{A} \oplus \boldsymbol{P}\| = \boldsymbol{x},\ \|\ominus \boldsymbol{B} \oplus \boldsymbol{P}\| = \boldsymbol{y},\ \|\ominus \boldsymbol{C} \oplus \boldsymbol{P}\| = \boldsymbol{z},\ \|\ominus \boldsymbol{D} \oplus \boldsymbol{P}\| = \boldsymbol{t}. \qquad (4.3)$$

Then by the gyrocosine theorem, Theorem 2.10. and by the Einstein-Pythagoras identity, Theorem 2.12. a point $\boldsymbol{D'}$ can be determine such that the gyrotriangles $\boldsymbol{D'PD}$ and $\boldsymbol{DPA}$ congruent, and the gyroline segment $\boldsymbol{AD'}$ is perpendicular to the gyrosegments $\boldsymbol{D'D}$ and and $\boldsymbol{AP}$. Thus we get a Saccheri gyroquadrilateral $\boldsymbol{APDD'}$ on the base $\boldsymbol{AD'}$, with base gyrolength $\|\ominus \boldsymbol{A} \oplus \boldsymbol{P}\|$, sides gyrolength x, gyrodiagonal gyrolength a, and summit gyrolegth t.

On the other hand from Theorem 2.12 and Theorem 3.1, for the Saccheri gyroquadriteral $\boldsymbol{ABCD}$ we have

$$\gamma_d = \frac{\gamma_c - 1 + \gamma_a^2}{\gamma_a} \qquad (4.4)$$

Hence by the (4.4), for the Saccheri gyroquadriteral $\boldsymbol{APDD'}$

$$\gamma_a = \frac{\gamma_t - 1 + \gamma_x^2}{\gamma_x} \tag{4.5}$$

Similary, we can obtain Saccheri gyroquadriterals $\boldsymbol{AA'BP}$, $\boldsymbol{BB'CP}$ and $\boldsymbol{CC'DP}$**.** Hence we can apply Theorem 3.5. to these Saccheri gyroquadriterals one by one. Then we have, respectively,

$$\gamma_b = \frac{\gamma_y - 1 + \gamma_x^2}{\gamma_x}, \tag{4.6}$$

$$\gamma_a = \frac{\gamma_t - 1 + \gamma_y^2}{\gamma_y}, \tag{4.7}$$

$$\gamma_c = \frac{\gamma_t - 1 + \gamma_z^2}{\gamma_z}. \tag{4.8}$$

Thus from (4.5) and (4.7), we obtain that

$$\gamma_t\gamma_y - \gamma_x\gamma_z = (1 + \gamma_x\gamma_y)(\gamma_y - \gamma_x) \tag{4.9}$$

Finally, from (4.4) we have

$$\gamma_z(\gamma_t - 1 + \gamma_x^2)^2 = \gamma_x^3(\gamma_t - 1 + \gamma_z^2 - \gamma_z). \tag{4.10}$$

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