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| RESEARCH ARTICLE

Approximation by Bivariate Half-Bernstein Operators on the Rectangular Domain

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| ABSTRACT

The problem of approximating a function in two dimensions is a central topic in approximation theory. This subject has played a major role in approximation. Many modifications of the classical Bernstein operator in two dimensions are introduced and studied to obtain an operator with a higher approximation order or a faster approximation rate. This paper defined the bivariate Bernstein operators by taking the half terms (even terms) to approximate a function in the space $C\big((0,1)\times(0,1)\big)$. Convergence was verified using Korovkin's theorem, while the asymptotic behavior was described using a Voronovskaja-type formula. Quantitative estimates for error are also derived using continuity measures suitable for bivariate functions. Also, the theoretical study is supported by a numerical example for a specified function in the space $C\big((0,1)\times(0,1)\big)$. A table and some graphs of the spacefied function and its approximations are plotted. The numerical example compares the performance of the proposed operator with that of the classical Bernstein operator in two dimensions. The numerical results showed that the operator has lower numerical approximation errors than the classical Bernstein operator in two dimensions. This work showed that the operator under study has a lower order of approximation than that of the classical Bernstein operator in two dimensions

| KEYWORDS

Bernstein operators, Voronovskaja-type asymptotics formula, Korovkin-type theorems

| ARTICLE INFORMATION

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1. Introduction

The theory of approximating functions using sequences of positive linear operators has been a central topic in approximation theory throughout the past century. The well-known Bernstein operators were introduced in [1].

$$B_n(\Psi;\xi)=\sum_{i=0}^{n} d_{n,i}(\xi).\Psi\left(\frac{i}{n}\right),$$

where, $d_{n,i}(\xi)=\binom{n}{i}\xi^i(1-\xi)^{n-i}$.

The Bernstein operators were later generalized to the second dimension, and Butzer's [2] work was among the first to provide the classical formulation of two-variable Bernstein operators over the domain $[0,1]\times[0,1]$, Korovkin's theorem [3] provided a basic theoretical framework for proving the convergence of sequences of positive linear operators. Following this, interest in multivariable Bernstein operators grew. Micchelli [4] studied the numerical properties of multivariable Bernstein polynomials and proposed an efficient computational method that significantly reduces the cost of evaluating them in several variables. Zeng and Cheng [5] studied the approximation rates of Bernstein operators and provided accurate estimates of the convergence rate at the continuity points of the function. Goyal et al. [6] developed bivariate Bernstein–Durrmeyer operators, and Pan et al. Çiçek et al. [7] studied a class of Bernstein operators based on several shape parameters. They showed that these operators provide greater flexibility and improve approximation accuracy compared to classical Bernstein operators. Gairola [8] introduced a new class of binary Bernstein operators and studied their approximation properties. A significant trend in this context is the study of half-Bernstein operators, which use only even indices and are appropriately normalized. These operators have been shown to retain linearity and positivity while achieving better numerical performance than classical Bernstein operators [9]. Recently, a half-modification of the Baskakov–Kantorovich sequence was introduced, using only half the terms of the original sequence. Despite this progress, most available results for half operators are limited to either univariate or enclosed domains. The generalization of these operators to bivariate operators on the open interval, particularly on the open square $(0,1) \times (0,1)$, has not been studied. Working with open domains is more analytically complex, as uniform convergence is not usually possible and requires reliance on point convergence and local uniform convergence techniques on the open interval. Building on recent developments, this paper presents a class of bivariate half-Bernstein operators on $(0,1) \times (0,1)$ such that $\xi, \tau \in (0,1) \times (0,1), \Psi \in C((0,1) \times (0,1))$ define as:

$$\mathcal{M}_n(\Psi; 2, \xi, \tau) = 4 \sum_{i=0}^{n} d_{2n,2i}(\xi) \left(\frac{i}{n}\right) \sum_{j=0}^{n} d_{2n,2j}(\tau) \left(\frac{j}{n}\right)$$

where, $d_{2n,2i}(\xi) d_{2n,2j}(\tau) = \binom{n}{i} \xi^{2i}(1-\xi)^{2n-2i} \binom{n}{j} \tau^{2j}(1-\tau)^{2n-2j}, (\xi, \tau) \in (0,1) \times (0,1).$

2. Preliminary Results

Some definitions and concepts are presented here to lay the foundation for the paper's main findings.

Lemma 2.1 For the function $d_{2n,2i}(\xi) d_{2n,2j}(\tau)$ The following fact holds

$$\xi(1-\xi) d'_{2n,2i}(\xi) d_{2n,2j}(\tau) = 2n \left(\frac{i}{n} - \xi\right) d_{2n,2i}(\xi) d_{2n,2j}(\tau)$$

Proof: We differentiate the function. $d_{2n,2i}(\xi) d_{2n,2j}(\tau)$, with respect to ξ, then multiply both ends by the expression $\xi(1 - \xi)$ to obtain

$$\xi(1-\xi) d'_{2n,2i}(\xi) d_{2n,2j}(\tau) = 2n\left(\frac{i}{n} - \xi\right) \binom{2n}{2i} \xi^{2i}(1-\xi)^{2n-2i} \binom{2n}{2j} \tau^{2j}(1-\tau)^{2n-2j})$$

$$= 2n \left(\frac{i}{n} - \xi\right) d_{2n,2i}(\xi) d_{2n,2j}(\tau).$$

Similarly, we differentiate the function with respect to τ, then multiply both sides of the equation by $\tau(1-\tau)$ to obtain:

$$\tau(1-\tau) d_{2n,2i}(\xi) d'_{2n,2j}(\tau) = 2n \left(\frac{j}{n} - \tau\right) d_{2n,2i}(\xi) d_{2n,2j}(\tau). \blacksquare$$

Definition 2.1. For $\alpha, \beta \in \mathbb{N}^0$, the central moments function of the sequence $\mathcal{M}_n(h; 2, \xi, \tau)$, defined as:

$$\wp_{2n,\alpha,\beta}(\xi,\tau) = \mathcal{M}_n\big((t_1-\xi)^\alpha (t_2-\tau)^\beta; 2, \xi, \tau\big) = 4 \sum_{i=0}^{n} d_{2n,2i}(\xi) \left(\frac{i}{n} - \xi\right)^\alpha \sum_{j=0}^{n} d_{2n,2j}(\tau) \left(\frac{j}{n} - \tau\right)^\beta.$$

Lemma 2.2. For $(\xi, \tau) \in (0,1) \times (0,1)$, and $\wp_{2n,\alpha,\beta}(\xi,\tau)$ has the following properties:

i) $\wp_{n,0,0}(\xi,\tau) = 1 + (1-2\xi)^{2n} + (1-2\tau)^{2n} + (1-2\xi)^{2n}(1-2\tau)^{2n}$

ii) $$\wp_{n,1,0}(\xi,\tau) = \mathcal{M}_n((t_1-\xi)^1 . \mathcal{M}_n(t_2-\tau)^0; 2, \xi, \tau)$$
$$= -2\xi(1-\xi)(1-2\xi)^{2n-1}(1+(1-2\tau)^{2n})$$

iii) $$\wp_{n,0,1}(\xi,\tau) = \mathcal{M}_n((t_1-\xi)^0 . \mathcal{M}_n(t_2-\tau)^1; 2, \xi, \tau)$$
$$= -2\tau(1-\tau)(1-2\tau)^{2n-1}(1+(1-2\xi)^{2n})$$

iv) $$\wp_{n,1,1}(\xi,\tau) = \mathcal{M}_n((t_1-\xi)^1 (t_2-\tau)^1; 2, \xi, \tau)$$
$$= -2\xi(1-\xi)(1-2\xi)^{2n-1}(1+(1-2\tau)^{2n}) - 2\tau(1-\tau)(1-2\tau)^{2n-1}(1+(1-2\xi)^{2n})$$

v) $$\wp_{n,2,0}(\xi,\tau) = \mathcal{M}_n((t_1-\xi)^2 . (t_2-\tau)^0; 2, \xi, \tau)$$
$$= \frac{\xi(1-\xi)}{2n}(1+(1-2\tau)^{2n})[1-(1-2\xi)^{2n} + 4(2n-1)\xi(1-\xi)(1-2\xi)^{2n-2}]$$

vi) $$\wp_{n,0,2}(\xi,\tau) = \mathcal{M}_n((t_1-\xi)^0 (t_2-\tau)^2; 2, \xi, \tau)$$
$$= \frac{\tau(1-\tau)}{2n}(1+(1-2\xi)^{2n})[1-(1-2\tau)^{2n} + 4(2n-1)\tau(1-\tau)(1-2\tau)^{2n-2}].$$

Proof:

i) $$\wp_{n,0,0}(\xi,\tau) = 4 \textstyle\sum_{i=0}^{n} d_{2n,2i}(\xi) \left(\frac{i}{n} - \xi\right)^0 \sum_{j=0}^{n} d_{2n,2j}(\tau) \left(\frac{j}{n} - \tau\right)^0$$
$$= 4 \sum_{i=0}^{n} d_{2n,2i}(\xi) \sum_{j=0}^{n} d_{2n,2j}(\tau) = 4 . \frac{(1+(1-2\xi)^n)}{2} . \frac{(1+(1-2\tau)^n)}{2}$$

$$= 1 + (1-2\xi)^{2n} + (1-2\tau)^{2n} + (1-2\xi)^{2n}(1-2\tau)^{2n}$$

ii) $\wp_{n,1,0}(\xi,\tau) = 4\sum_{i=0}^{n} d_{2n,2i}(\xi)\,(\frac{i}{n}-\xi)^1 \sum_{j=0}^{n} d_{2n,2j}(\tau)\,(\frac{j}{n}-\tau)^0$

$$\wp_{n,1,0}(\xi,\tau) = 4\sum_{i=0}^{n} d_{2n,2i}(\xi)\,(\frac{i}{n}-\xi)^1 \sum_{j=0}^{n} d_{2n,2j}(\tau)$$

$$= 4(\sum_{i=0}^{n} b_{2n,2i}(\xi)\left(\frac{i}{n}\right) - \xi\sum_{j=0}^{n} b_{2n,2j}(\xi))(\sum_{j=0}^{n} b_{2n,2j}(\tau))$$

$$= 4(-\frac{\xi(1-2\xi)^{2n-1}}{2}\big[\big(1+(1-2\xi)\big)\big](\frac{(1+(1-2\tau)^{2n})}{2})$$

$$= -\xi(1-2\xi)^{2n-1}[2-2\xi](1+(1-2\tau)^{2n})$$

$$= -2\xi[1-\xi](1-2\xi)^{2n-1}(1+(1-2\tau)^{2n}).$$

In the sème way, we proved

iii) $\wp_{n,0,1}(\xi,\tau) = -2\tau(1-\tau)(1-2\tau)^{2n-1}(1+(1-2\xi)^{2n})$.

iv) Using (ii) and (iii), we prove that

$$\wp_{n,1,1}(\xi,\tau) = 4\sum_{i=0}^{n} d_{2n,2i}(\xi)\,(\frac{i}{n}-\xi)^1 \sum_{j=0}^{n} d_{2n,2j}(\tau)\,(\frac{j}{n}-\tau)^1$$

$$= (-2\xi(1-\xi)(1-2\xi)^{2n-1}(1+(1-2\tau)^{2n}))$$

$$(-2\tau(1-\tau)(1-2\tau)^{2n-1}(1+(1-2\xi)^{2n}))$$

v) $\wp_{n,2,0}(\xi,\tau) = 4\sum_{i=0}^{n} d_{2n,2i}(\xi)\,(\frac{i}{n}-\xi)^2 \sum_{j=0}^{n} d_{2n,2j}(\tau)\,(\frac{j}{n}-\tau)^0$

$$\wp_{n,2,0}(\xi,\tau) = (4\sum_{i=0}^{n} d_{2n,2i}(\xi)\,(\frac{i^2}{n^2}) - 8\xi\sum_{i=0}^{n} d_{2n,2i}(\xi)\,(\frac{i}{n}) + 4\xi^2\sum_{i=0}^{n} d_{2n,2i}(\xi))\sum_{j=0}^{n} d_{2n,2j}(\tau)$$

$$= \frac{\xi^2(2n-1)}{n}\left\{\frac{(1+(1-2\xi)^{2n-2}}{2}\right\} + \frac{\xi}{n}\left[\frac{(1-(1-2\xi)^{2n-1}}{2}\right] - 2\xi^2(1-(1-2\xi)^{2n-1}) + \xi^2(1+(1-2\xi)^{2n}))((1+(1-2\tau)^{2n}).$$

vi) $\wp_{n,0,2}(\xi,\tau) = 4\sum_{i=0}^{n} d_{2n,2i}(\xi)\,(\frac{i}{n}-\xi)^0 \sum_{j=0}^{n} d_{2n,2j}(\tau)\,(\frac{j}{n}-\tau)^2$

$$= \frac{\tau^2(2n-1)}{n}\left\{\frac{(1+(1-2\tau)^{2n-2}}{2}\right\} + \frac{\tau}{n}\left[\frac{(1-(1-2\tau)^{2n-1}}{2}\right] - 2\tau^2(1-(1-2\tau)^{2n-1}) + \tau^2(1+(1-2\tau)^{2n}))((1+(1-2\xi)^{2n}).\ \blacksquare$$

Now, $\wp_{n,\alpha,\beta}$ has the following recurrence relations:

$$2n\wp_{n,\alpha+1,\beta}(\xi,\tau) = \xi(1-\xi)\left[\alpha\,\wp_{n,\alpha-1,\beta}(\xi,\tau) + \frac{\partial}{\partial\xi}\wp_{n,\alpha,\beta}(\xi,\tau)\right], \alpha \geq 2$$

$$2n\wp_{n,\alpha,\beta+1}(\xi,\tau) = \tau(1-\tau)\left[\alpha\,\wp_{n,\alpha,\beta-1}(\xi,\tau) + \frac{\partial}{\partial\tau}\wp_{n,\alpha,\beta}(\xi,\tau)\right], \beta \geq 2.$$

Lemma 2.3. For $(\xi,\tau) \in (0,1)\times(0,1)$, and $\lim_{n\to\infty} 2n\,\wp_{n,\alpha,\beta}(\xi,\tau)$ has the following properties:

i) $\lim_{n\to\infty} 2n\,\wp_{n,0,0}(\xi,\tau) = 0$,

ii) $\lim_{n\to\infty} 2n\,\wp_{n,1,0}(\xi,\tau) = 0$,

iii) $\lim_{n\to\infty} 2n\,\wp_{n,0,1}(\xi,\tau) = 0$.

iv) $\lim_{n\to\infty} 2n\,\wp_{n,1,1}(\xi,\tau) = 0$.

v) $\lim_{n\to\infty} 2n\,\wp_{n,2,0}(\xi,\tau) = \xi(1-\xi)$.

vi) $\lim_{n\to\infty} 2n\,\wp_{n,0,2}(\xi,\tau) = \tau(1-\tau)$.

3. Main Results :

This section présents the asymptotic behavior and convergence rates of the modified bivariate operators. $M_n\,(h;2,\xi,\tau)$ by developing a Voronovskaya-type theory on open intervalles. The exact ordre of approximation is détermine by analyzing the boundary behavior of the normalized différences. In addition, quantitative errer estimâtes of the approximation procès are presented using continuité paramètre techniques adapte to the réguler local convergence Framework on $(0,1)\times(0,1)$

Theorem 3.1. The sequence $\mathcal{M}_n(\Psi;2,\xi,\tau),(\xi,\tau)\in(0,1)\times(0,1)$, is converges to the function $\Psi(\xi,\tau)$ as $n\to\infty,\Psi\in C((0,1)\times(0,1))$

Proof.: By using the Korovkin theorem [10], and from the fact that $\forall(\xi,\tau)\in(0,1)\times(0,1)\to|1-2\xi|<1$, and $|1-2\tau|<1$, therefore; $\lim\limits_{n\to\infty}[(1-2\xi)^{2n}+(1-2\tau)^{2n}+(1-2\xi)^{2n}(1-2\tau)^{2n}]\to 0$ as $n\to\infty$, the following conditions hold:

i) $\mathcal{M}_n(1;2,\xi,\tau)=4\ \sum_{i=0}^{n}d_{2n,2i}(\xi)\sum_{j=0}^{n}d_{2n,2j}(\tau)\ =4.\frac{(1+(1-2\xi)^n)}{2}.\frac{(1+(1-2\tau)^n)}{2}$

$$\mathcal{M}_n(1;2,\xi,\tau)\ \to\ 1\ \text{ as } n\to\infty.$$

ii) $\mathcal{M}_n(t;2,\xi,\tau)=4\ \sum_{i=0}^{n}d_{2n,2i}(\xi)\,(\frac{i}{n})\sum_{j=0}^{n}d_{2n,2j}(\tau)$

$$=4(\sum_{i=0}^{n}\binom{2n}{2i}\xi^{2i}(1-\xi)^{2n-2i}.(\frac{i}{n}))\,(\sum_{j=0}^{n}\binom{2n}{2j}\tau^{2j}(1-\tau)^{2n-2j})$$

$$\mathcal{M}_n(t;2,\xi,\tau)=2\xi.\frac{(1-(1-2\xi)^{2n-1})}{2}(1+(1-2\tau)^{2n}).$$

$$\mathcal{M}_n(t;2,\xi,\tau)=\xi(1-(1-2\xi)^{2n-1})(1+(1-2\tau)^{2n}).$$

$$\mathcal{M}_n(t;2,\xi,\tau)\to\xi\ \text{ as }\ n\to\infty\,,\forall(\xi,\tau)\in(0,1)\times(0,1).$$

In the same way, we proved

iii) $\mathcal{M}_n(s;2,\xi,\tau)=\tau(1-(1-2\xi)^{2n})(1+(1-2\tau)^{2n-1})\ \to\tau,\ \text{ as }\ n\to\infty$

Using (ii) and (iii), we prove that

iv) $\mathcal{M}_n(ts;2,\xi,\tau)\to\xi\tau,\ \text{ as }\ n\to\infty,\forall(\xi,\tau)\in(0,1)\times(0,1).$

v) $\mathcal{M}_n(t^2;2,\xi,\tau)=4\ \sum_{i=0}^{n}d_{2n,2i}(\xi)\,(\frac{i}{n})^2\sum_{j=0}^{n}d_{2n,2j}(\tau)$

$$=4(\sum_{i=0}^{n}\binom{2n}{2i}\xi^{2i}(1-\xi)^{2n-2i}.(\frac{i}{n})^2)\,(\sum_{j=0}^{n}\binom{2n}{2j}\tau^{2j}(1-\tau)^{2n-2j})$$

$$=\frac{2}{n^2}(\sum_{i=0}^{n}i^2\binom{2n}{2i}\xi^{2i}(1-\xi)^{2n-2i}).2(\frac{(1+(1-2\tau)^{2n})}{2})$$

$$=\frac{\xi^2(2n-1)}{2n}(1+(1-2\xi)^{2n-2}+\frac{\xi}{2n}(1-(1-2\xi)^{2n-1}).((1+(1-2\tau)^{2n})$$

$\mathcal{M}_n(t^2;2,\xi,\tau)\to\xi^2\ \text{ as }\ n\to\infty,\forall(\xi,\tau)\in(0,1)\times(0,1).$

Using a proof similar to the previous one, we obtain:

vi) $\mathcal{M}_n(s^2;2,\xi,\tau)=\frac{\tau^2(2n-1)}{2n}(1+(1-2\tau)^{2n-2}+\frac{\tau}{2n}(1-(1-2\tau)^{2n-1}).((1+(1-2\xi)^{2n})$

$$\mathcal{M}_n(s^2;2,\xi,\tau)\to\xi^2,\ \text{ as }\ n\to\infty.$$

Hence, $\mathcal{M}_n(\Psi;2,\xi,\tau)\to\Psi(\xi,\tau)$ as $n\to\infty$. ■

Theorem 3.2. For $(\xi,\tau)\in(0,1)\times(0,1),\Psi\in C\big((0,1)\times(0,1)\big)$, if $\frac{\partial^2\Psi}{\partial\xi^2}$ and $\frac{\partial^2\Psi}{\partial\tau^2}$ If they exist and are continuous, then the following result is correct.

$$\lim_{n\to\infty}2n[\mathcal{M}_n(\Psi;2,\xi,\tau)-\Psi(\xi,\tau)]=\frac{\xi(1-\xi)}{2}\,\frac{\partial^2\Psi(\xi,\tau)}{\partial\xi^2}+\frac{\tau(1-\tau)}{2}\,\frac{\partial^2\Psi(\xi,\tau)}{\partial\tau^2}$$

Proof: By Taylor's expansion [11] of $\Psi(t,s)$ at the point $t,s,\xi,\tau\in(0,1)\times(0,1)$ is provided as follows:

$$\Psi(t,s)=\Psi(\xi,\tau)+\frac{\partial\Psi(\xi,\tau)}{\partial\xi}(t-\xi)+\frac{\partial\Psi(\xi,\tau)}{\partial\tau}(s-\tau)+\frac{1}{2}\frac{\partial^2\Psi(\xi,\tau)}{\partial\xi^2}\,(t-\xi)^2+\frac{1}{2}\frac{\partial^2\Psi(\xi,\tau)}{\partial\tau^2}\,(s-\tau)^2+\frac{\partial^2\Psi(\xi,\tau)}{\partial\xi\partial\tau}\big((t-\xi)(s-\tau)\big)$$
$$+\,\emptyset(t,s;\xi,\tau)\sqrt{(t-\xi)^4+(s-\tau)^4},$$

where $\emptyset(t,s;\xi,\tau)=o((t-\xi)^2+(s-\tau)^2\to 0$ as $(t,s)\to(\xi,\tau)$.

Apply the linear positive operator and subtract $\Psi(\xi,\tau)$to both sides, and using Lemma 2.2, we get:

$$\mathcal{M}_n(\Psi;2,\xi,\tau)-\Psi(\xi,\tau)$$
$$=\Psi(\xi,\tau)\big[\wp_{n,0,0}-1\big]+\frac{\partial\Psi(\xi,\tau)}{\partial\xi}\,\wp_{n,1,0}+\frac{\partial\Psi(\xi,\tau)}{\partial\tau}\,\varphi_{n,0,1}+\frac{1}{2}\frac{\partial^2\Psi(\xi,\tau)}{\partial\xi^2}\,\wp_{n,2,0}+\frac{1}{2}\frac{\partial^2\Psi(\xi,\tau)}{\partial\tau^2}\wp_{n,0,2}$$
$$+\frac{\partial^2\Psi(\xi,\tau)}{\partial\xi\partial\tau}\,\wp_{n,1,1}+\mathcal{M}_n(\emptyset(t,s;\xi,\tau)\sqrt{(t-\xi)^4+(s-\tau)^4};2,\xi,\tau)$$

We multiply both sides by (2n) and take $\lim\limits_{n\to\infty}$ Then, using Lemma 2.3, we get:

$$\lim_{n\to\infty} 2n[\mathcal{M}_n(\Psi,2,\xi,\tau)-\Psi(\xi,\tau)] = \frac{\xi(1-\xi)}{2}\frac{\partial^2\Psi(\xi,\tau)}{\partial\xi^2}+\frac{\tau(1-\tau)}{2}\frac{\partial^2\Psi(\xi,\tau)}{\partial\tau^2}+\lim_{n\to\infty}\left(2n\mathcal{M}_n(\emptyset(t,s;\xi,\tau)\sqrt{(t-\xi)^4+(s-\tau)^4};2,\xi,\tau)\right).$$

Now, shows that the remainder term tends to zero $\lim_{n\to\infty}\left(2n\,\mathcal{M}_n(\emptyset(t,s;\xi,\tau)\sqrt{(t-u)^4+(s-w)^4};2,\xi,\tau)\right)=0$

By the Cauchy-Schwarz inequality [12], and linearity$\mathcal{M}_n(\Psi;2,\xi,\tau)$, we have

$$\mathcal{M}_n\left((\emptyset(t,s;\xi,\tau)\sqrt{(t-\xi)^4+(s-\tau)^4};2,\xi,\tau)\right)\le(\mathcal{M}_n(\emptyset^2(t,s;\xi,\tau)\,;2,\xi,\tau)^{\frac{1}{2}}))(\mathcal{M}_n((t-\xi)^4+(y-\tau)^4;2,\xi,\tau))^{\frac{1}{2}}$$

Since $\emptyset^2(t,s;\xi,\tau)\to 0$ as $(t,s)\to(\xi,\tau)$. And $\mathcal{M}_n$ is positive and convergent , we obtain

$$\mathcal{M}_n(\emptyset^2(t,s;\xi,\tau)\,;2,\xi,\tau)\to 0$$

Also, By using the recurrence relations from Lemma 2.2, we can compute these fourth-order moments. $\mathcal{M}_n((t-\xi)^4;2,\xi,\tau))=o(\frac{1}{n^2}),\mathcal{M}_n((s-\tau)^4;2,\xi,\tau))=o(\frac{1}{n^2})$

Hence $(2n\mathcal{M}_n((t-\xi)^4+(s-\tau)^4;2,\xi,\tau))^{\frac{1}{2}}=o(\frac{1}{n})$

Consequently, $\lim_{n\to\infty}\left(2n\mathcal{M}_n(\emptyset(t,s;\xi,\tau)\sqrt{(t-\xi)^4+(s-\tau)^4};2,\xi,\tau)\right)\to 0$

Therefore, $\lim_{n\to\infty} 2n[\mathcal{M}_n(\Psi,2,\xi,\tau)-\Psi(\xi,\tau)]=\frac{\xi(1-\xi)}{2}\frac{\partial^2\Psi(\xi,\tau)}{\partial\xi^2}+\frac{\tau(1-\tau)}{2}\frac{\partial^2\Psi(\xi,\tau)}{\partial\tau^2}$. ■

Next, we analyze the convergence rate of the operator. $M_n(\Psi;\ 2,\xi,\tau)$ to the function $\Psi(\xi,\tau)$. We begin by defining some smoothness measures for bivariable functions, based on reference [13], for the functions $\Psi\in C((0,1)\times(0,1))$

Complete modulus of continuity

$$\omega(\Psi;\delta_1,\delta_2)=sup\{|\Psi(\xi_1,\tau_1)-\Psi(\xi_2,\tau_2)|:(\xi_1,\tau_1),(\xi_2,\tau_2)\in(0,1)\times(0,1),|\xi_1-\xi_2|\le\delta_1,|\tau_1-\tau_2|\le\delta_2\},$$

Partial modulus of continuity

$$\omega_\xi(\Psi;\delta_1)=sup\{sup\{|\Psi(\xi_1,\tau)-\Psi(\xi_2,\tau)|:|\xi_1-\xi_2|\le\delta_1\},(\xi_1,\tau),(\xi_2,\tau)\in(0,1)\},$$
$$\omega_\tau(\Psi;\delta_2)=\sup\{\sup\{|\Psi(\xi,\tau_1)-\Psi(\xi,\tau_2)|:|\tau_1-\tau_2|\le\delta_2\},(\xi,\tau_1),(\xi,\tau_2)\in(0,1)\},$$

Where $\delta_1>0,\delta_2>0,\delta>0$. The following inequalities hold for the aforementioned smoothness modulus [13]:

$\omega_\xi(\Psi;\gamma\delta_1)\le(1+\gamma)\omega_\xi(\Psi;\delta_1)$, $\omega_\tau(\Psi;\gamma\delta_2)\le(1+\gamma)\omega_\tau(\Psi;\delta_2);\gamma\ge 0;$

Theorem 3.3. For $\Psi\in C((0,1)\times(0,1))$, the following holds $|\mathcal{M}_n(\Psi;2,\xi,\tau)|\le\|\Psi\|+o(1)$
where $\|\Psi\|=\sup_{(s,t)}|\Psi(s,t)|$

Proof: $|\mathcal{M}_n(\Psi;2,\xi,\tau)|=|4\sum_{i=0}^{n}\sum_{j=0}^{n}d(2n,2i,2j,\xi,\tau).\Psi\left(\frac{i}{n},\frac{j}{n}\right)|$

$$\le 4\sum_{i=0}^{n}\sum_{j=0}^{n}d(2n,2i,2j,\xi,\tau).\left|\Psi\left(\frac{i}{n},\frac{j}{n}\right)\right|$$
$$\le\|\Psi\|.\,4\sum_{i=0}^{n}\sum_{j=0}^{n}d(2n,2i,2j,\xi,\tau)$$
$$=\|\Psi\|.\,\mathcal{M}_n(1;2,\xi,\tau)$$
$$\mathcal{M}_n(1;2,\xi,\tau)=(1+(1-2\xi)^{2n})(1+(1-2\tau)^{2n})=\wp_{n,0,0}(\xi,\tau)$$
$$\le\wp_{n,0,0}(\xi,\tau).\,\|\Psi\|=\|\Psi\|+o(1).$$ ■

Theorem 3.4. If $\Psi\in C(0,1)^2,(\xi,\tau)\in(0,1)\times(0,1)$then
$|\mathcal{M}_n(\Psi;2,\xi,\tau)-\Psi(\xi,\tau)|\le 4\omega_{\delta 1,\delta_2}(\Psi)+o(1)$.

Proof: By applying the property of the moulus of continuité $(s,t),(u,w)\in(0,1)\times(0,1)$[10], thn

$$|\Psi(t,s)-\Psi(\xi,\tau)|\le\omega_{\delta 1,\delta_2}(\Psi)(1+\frac{|t-\xi|}{\delta_1})(1+\frac{|s-\tau|}{\delta_2})$$
$$\omega_{\delta 1,\delta_2,}(\Psi)=sup_{\substack{|t-\xi|\le\delta_1\\|s-\tau|\le\delta_2}}|\Psi(t,s)-\Psi(\xi,\tau)|$$

Now, by taking the sequence $\mathcal{M}_n$ for both sides

$$|\mathcal{M}_n(\Psi;2,\xi,\tau)-\Psi(\xi,\tau)|\le\omega_{\delta 1,\delta_2}(\Psi)(\mathcal{M}_n(1;2,\xi,\tau)+\mathcal{M}_n\left(\frac{|t-\xi|}{\delta_1};2,\xi,\tau\right)+\mathcal{M}_n(\frac{|s-\tau|}{\delta_2};2,\xi,\tau)+\mathcal{M}_n\left(\frac{|t-\xi||s-\tau|}{\delta_1\delta_2};2,\xi,\tau\right))$$

Using the Cauchy-Schwarz inequality [12] and Lemma 2.2, one has

$$=\omega_{\delta 1,\delta_2}(\Psi)\{\wp_{n,0,0}(\xi,\tau)+(\frac{1}{\delta 1}\left(\wp_{n,2,0}(\xi,\tau).\wp_{n,0,0}(\xi,\tau)\right)^{\frac{1}{2}}+\left(\frac{1}{\delta 2}\left(\wp_{n,0,2}(\xi,\tau).\wp_{n,0,0}(\xi,\tau)\right)^{\frac{1}{2}}\right)$$
$$+\frac{1}{\delta 1\delta 2}\left(\wp_{n,2,0}(\xi,\tau).\wp_{n,0,0}(\xi,\tau)\right)^{\frac{1}{2}}\left(\wp_{n,0,2}(\xi,\tau).\wp_{n,0,0}(\xi,\tau)\right)^{\frac{1}{2}}\}$$

We choose,$\delta 1=\sqrt{\wp_{n,2,0}(\xi,\tau)\wp_{n,0,2}(\xi,\tau)}$, $\delta 2=\sqrt{\wp_{n,2,0}(\xi,\tau)\wp_{n,0,0}(\xi,\tau)}$

$$=\omega_{\delta 1,\delta_2}(\Psi)(\wp_{n,0,0}(\xi,\tau)+1+1+1)$$
$$=\omega_{\delta 1,\delta_2}(\Psi)(4+o(1))$$

Hence, $|\mathcal{M}_n(\Psi;2,\xi,\tau)-\Psi(\xi,\tau)|\le 4\omega_{\delta 1,\delta_2}(\Psi)+o(1)$. ■

Theorem 3.4. If $\Psi \in C((0,1)\times(0,1)), (\xi,\tau) \in (0,1)\times(0,1)$then
$|\mathcal{M}_n(\Psi;2,\xi,\tau)-\Psi(\xi,\tau)| \le (2+o(1))\{\omega_\xi(\Psi;\sqrt{\wp_{n,2,0}(\xi,\tau)})+\omega_\tau(\Psi;\sqrt{\wp_{n,2,0}(\xi,\tau)})\}$.

Proof: $|\Psi(t,s)-\Psi(\xi,\tau)| = |\Psi(t,s)-\Psi(\xi,s)+\Psi(\xi,s)-\Psi(\xi,\tau)|$

$$|\Psi(t,s)-\Psi(\xi,\tau)| \le |\Psi(t,s)-\Psi(\xi,s)|+|\Psi(\xi,s)-\Psi(\xi,\tau)|$$

By applying the property of the partial modulus of continuity $(s,t),(\xi,\tau) \in (0,1)\times(0,1)$, then

$$|\Psi(t,s)-\Psi(\xi,\tau)| \le (1+\frac{|t-\xi|}{\delta_1})\omega_\xi(\Psi;\delta_1)+(1+\frac{|s-\tau|}{\delta_2})\omega_\tau(\Psi;\delta_2)$$

Now, by taking the sequence $\mathcal{M}_n$ for both sides

$$|\mathcal{M}_n(\Psi;2,\xi,\tau)-\Psi(\xi,\tau)| \le \omega_\xi(\Psi;\delta_1)\left\{\mathcal{M}_n(1;2,\xi,\tau)+\frac{1}{\delta_1}\mathcal{M}_n(|t-\xi|;2,\xi,\tau)\right\}+\omega_\tau(\Psi;\delta_2)\left\{\mathcal{M}_n(1;2,\xi,\tau)+\frac{1}{\delta_2}\mathcal{M}_n(|s-\tau|;2,\xi,\tau)\right\}$$

Using the Cauchy-Schwarz inequality [9] and Lemma 2.2

$$\mathcal{M}_n(|t-\xi|;2,\xi,\tau) \le \sqrt{\mathcal{M}_n((t-\xi)^2;2,\xi,\tau)\mathcal{M}_n(1;2,\xi,\tau)} = \sqrt{\wp_{n,2,0}(\xi,\tau)\wp_{n,0,0}(\xi,\tau)}$$

$$\mathcal{M}_n(|s-\tau|;2,\xi,\tau) \le \sqrt{\mathcal{M}_n((s-\tau)^2;2,\xi,\tau)\mathcal{M}_n(1;2,\xi,\tau)} = \sqrt{\wp_{n,0,2}(\xi,\tau)\wp_{n,0,0}(\xi,\tau)}$$

We choose, $\delta 1 = \sqrt{\wp_{n,2,0}(\xi,\tau)}$, and $\delta 1 = \sqrt{\wp_{n,0,2}(\xi,\tau)}$, and
$\because \mathcal{M}_n(1;2,\xi,\tau) = 1+o(1)$ we get

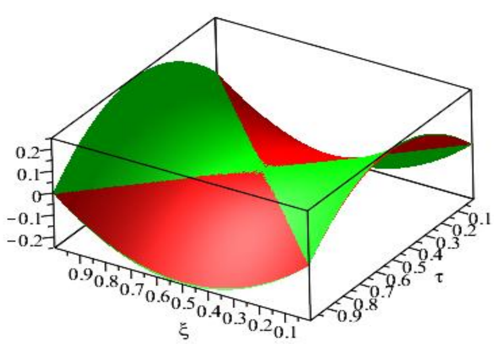
$$|\mathcal{M}_n(\Psi;2,\xi,\tau)-\Psi(\xi,\tau)| \le (2+o(1))\{\omega_\xi(\Psi;\delta_1)+\omega_\tau(\Psi;\delta_2)\}. \blacksquare$$

4. Numerical Examples

In this study, the approximation properties of the bivariate half-Bernstein sequence $M_n\ (\Psi,2,\xi,\tau)$ were studied for selected test functions. The numerical results show that the performance of the bivariate half-Bernstein sequence approximation is numerically superior to that of the conventional bivariate Bernstein séquence approximation.

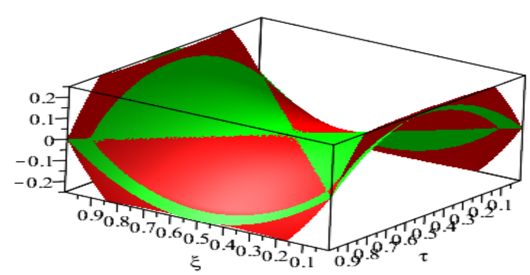
Example 4.1. We present a comparison of the convergence of $B_n(\Psi;\xi,\tau)$ and $\mathcal{M}_n(\Psi;2,\xi,\tau)$ for the test function $\Psi(\xi,\tau) = (\xi-\frac{1}{2})^2-\left(\tau-\frac{1}{2}\right)^2$, $n = 30{,}60$, and 100 on the interval (0,1), respectively. The error functions of these polynomials are also compared, and the results are presented in the accompanying figures.

Fig.1: The following figures compares the approximating B_n and M_n operators for the function $\Psi(\xi,\tau)$ at values of $n = 30, r = 2$.



- $\Psi(\xi,\tau)$ – $B_n(\Psi;30,\xi,\tau)$

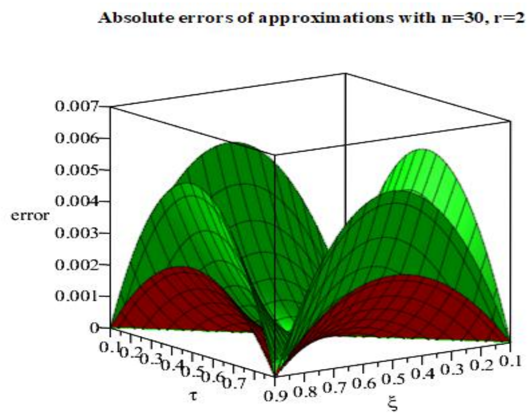
(a)



–$\Psi(\xi,\tau)$ – $\mathcal{M}_n(\Psi;30{,}2,\xi,\tau)$

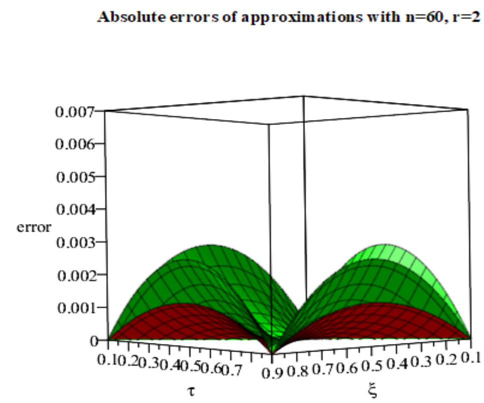
(b)

Fig.2: The following figure compares the average absolute errors in approximating the B_n and $\mathcal{M}_n$ operators for the function Ψ(ξ,τ) at two different values of n= 30 and 60.



Green color represents, $|\Psi(\xi,\tau) - B_n(\Psi; 30, \xi, \tau)|$,

and red represents, $|\Psi(\xi,\tau) - \mathcal{M}_n(\Psi; 30{,}2, \xi, \tau)|$



Green color represents, $|\Psi(\xi,\tau) - B_n(\Psi; 60, \xi, \tau)|$
, and red represents, $|\Psi(\xi,\tau) - \mathcal{M}_n(\Psi; 60{,}2, \xi, \tau)|$

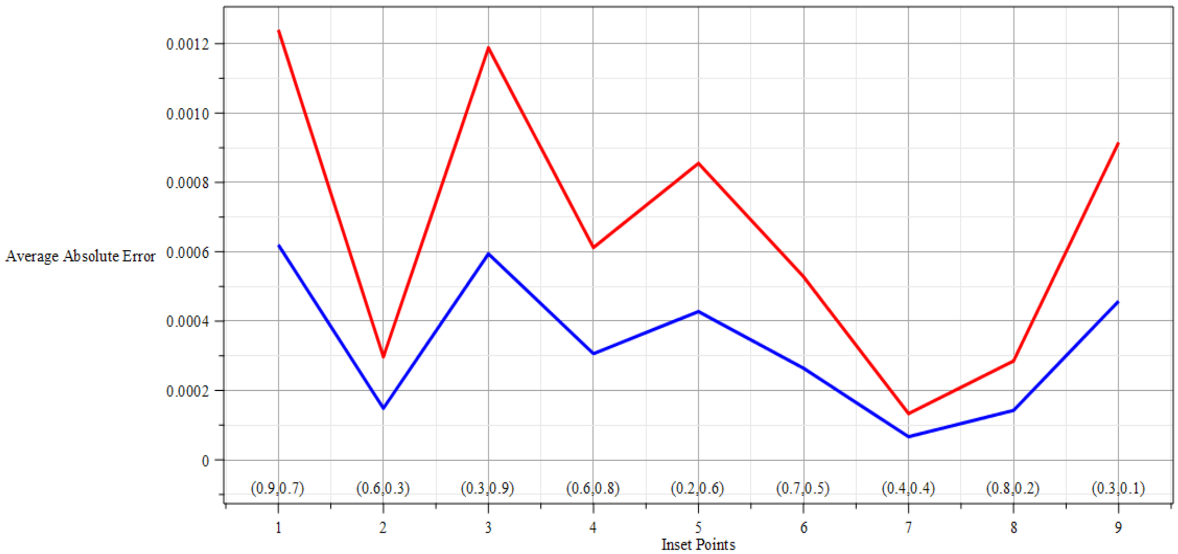
(a) **(b)**

The green area represents the error due to the classical Bernstein operator B_n, while the red area represents the error due to the half- Bernstein operator M_n. As shown in the figure, the red areas are smaller than the green areas in both cases, indicating that theM_n. operator provides a more accurate approximation than the conventional Bernstein operator. Furthermore, the error values decrease significantly from $n = 30$ to $n = 60$, confirming convergence and the improvement in approximation accuracy with increasing n. Therefore, the numerical and graphical results demonstrate the efficiency of the proposed M_n. operator and its superiority in reducing errors and achieving better approximation of the function under study.

Table 1. The table shows the average absolute errors of approximations using the operators.$B_n$and $\mathcal{M}_n$ at different values of n, for example, 4.1 with $m = 0.02$

n	$\frac{\sum_{i=1}^{s}\sum_{j=1}^{s}\left\|B_n\left(\Psi,\frac{i}{m},\frac{j}{m}\right)-\Psi(\frac{i}{m},\frac{j}{m})\right\|}{s^2}$	$\frac{\sum_{i=1}^{s}\sum_{j=1}^{s}\left\|\mathcal{M}_n\left(\Psi,2,\frac{i}{m},\frac{j}{m}\right)-\Psi(\frac{i}{m},\frac{j}{m})\right\|}{s^2}$
30	0.02139917695	0.01069776199
60	0.01069958848	0.00534979424
90	0.00713305898	0.00356652949
120	0.00534979424	0.00267489712
150	0.00427983539	0.00213991770

We observe that the average error values for both operators decrease gradually as n increases, confirming convergence and improving the approximation's accuracy. It is also clear that the average error values for the operator M_n are smaller than the average error values for the operator B_nfor all values of n considered, indicating that has higher efficiency and better accuracy in approximating the function Ψ. Therefore, the numerical results in the table show the superiority of the operator. M_n over the conventional Bernstein operator in terms of reducing the average error and improving the quality of the approximation.



The orange curve represents the errors of the classical bivariate Bernstein operator. B_n, while the blue curve represents the errors of the proposed bivariate half-Bernstein operator M_n . It can be observed that the bivariate half-Bernstein operator generally yields smaller approximation errors at most of the tested points, indicating improved approximation accuracy.

We define the improvement ratio for the operator M_n with respect to B_n as follows:

$$\mathrm{Imp(n)} = \frac{\mathrm{AE(B_n)} - \mathrm{AE}(\mathcal{M}_n)}{\mathrm{AE(B_n)}} \times 100\%$$

The numerical results indicate that $AE(\mathcal{M}_n) \approx \frac{1}{2}\, AE(B_n)$. This means achieving approximately 50% improvement, indicating a significant decrease in rounding error.

5. Conclusions :

This research introduces the bivariate half-Bernstein operator as a modified version of the bivariate classical Bernstein operator and examines its approximation properties. The convergence of the proposed operator towards continuous functions on the interval $(0,1) \times (0,1)$ is demonstrated using Korovkin's theorem. The Voronovskaya formula is derived, and the error is estimated using the continuity coefficient. The numerical and graphical results show that the bivariate half-Bernstein operators exhibit lower average absolute errors than the classical operators at various values of n in the domain $(0,1) \times (0,1)$, indicating an improvement in the approximation order. Therefore, the proposed operators can be considered an effective tool for improving the approximation of binary functions.

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