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| RESEARCH ARTICLE

On universality and simple a-points of a family of Dirichlet L-functions and their derivatives

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| ABSTRACT
In 1975, Voronin established the celebrated universality theorem for the Riemann zeta-function, and later, Bagchi and Gonek independently obtained its analogue for a certain family of Dirichlet L-functions in the q-aspect. In this paper, we generalize their results by establishing a simultaneous universality theorem for the family of Dirichlet L-functions and their derivatives in the q-aspect. Furthermore, as an application of this simultaneous universality, we investigate the value-distribution of these functions. Specifically, we show that for arbitrarily given complex numbers a_j and open disks in the right half of the critical strip, there exists a positive proportion of Dirichlet characters χ modulo a large prime q such that the Dirichlet L-function $L(s,\chi)$ and its derivatives simultaneously have simple a_j-points in the prescribed disks.

| KEYWORDS
Dirichlet L-function, derivatives, universality, a-point, value-distribution

| ARTICLE INFORMATION

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1. Introduction and statement of results

One of the most fascinating functions in mathematics is the Riemann zeta-function $\zeta(s)$. The study of value-distribution for $\zeta(s)$ has a long history (see [22, Chapter XI]). In 1914, Bohr and Courant [2] showed that the set $\{\zeta(\sigma_0+i\tau)\colon \tau\in\mathbb{R}\}$ is dense in $\mathbb{C}$ for any fixed real number σ_0 with $1/2<\sigma_0<1$. Extending this result, in 1972, Voronin [23] showed that if n is a positive integer then the set

$$\left\{\left(\zeta(\sigma_0+i\tau),\zeta^{(1)}(\sigma_0+i\tau),\dots,\zeta^{(n)}(\sigma_0+i\tau)\right)\in\mathbb{C}^{n+1}\colon \tau\in\mathbb{R}\right\}$$

is dense in $\mathbb{C}^{n+1}$ for any fixed real number σ_0 with $1/2<\sigma_0<1$. Here, $\zeta^{(j)}(s)$ denotes the j-th derivative of $\zeta(s)$.

Furthermore, in 1975, Voronin [24] obtained a remarkable result, which is called the *universality* theorem for $\zeta(s)$. His result above in 1972 is a corollary of this theorem (see [10, p. 252, Theorem 2]). This theorem asserts, roughly, that any nonvanishing analytic function can be uniformly approximated by vertical shifts of $\zeta(s)$. We now state it in the modern form, which is shown in [11, p. 225, Theorem 5.2]. For a measurable set $A\subset\mathbb{R}$, we denote the Lebesgue measure of A by $\operatorname{meas} A$.

Theorem A. *Let $\mathcal{K}$ be a compact subset of $\mathcal{D}:=\{s\in\mathbb{C}\colon 1/2<\operatorname{Re}s<1\}$ such that $\mathbb{C}\backslash\mathcal{K}$ is connected. Let $f(s)$ be a nonvanishing continuous function on $\mathcal{K}$ which is holomorphic in the interior (if any) of $\mathcal{K}$. Then for any $\epsilon>0$ we have*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau\in[0,T]\colon \max_{s\in\mathcal{K}}|\zeta(s+i\tau)-f(s)|<\epsilon\right\}>0.$$

Universality theorems have been established for various zeta and L-functions, and provide interesting consequences (see e.g. [14], [18] and [21]).

As in Theorem B below, a similar universality property holds for every derivative $\zeta^{(\nu)}(s)$. This theorem is shown in [12, Theorem 1 and p. 89, l. 4], [13] and [15, Theorem 2 and Remark (i)]. Notice that the target function $f(s)$ here is not necessarily nonvanishing. Let $\mathbb{N}$ denote the set of positive integers.

Theorem B. *Let $\nu \in \mathbb{N}$. Let $\mathcal{K}$ be a compact subset of $\mathcal{D}$ such that $\mathbb{C}\backslash\mathcal{K}$ is connected. Let $f(s)$ be a continuous function on $\mathcal{K}$ which is holomorphic in the interior (if any) of $\mathcal{K}$. Then for any $\epsilon > 0$ we have*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau \in [0,T]\colon \max_{s\in\mathcal{K}}\left|\zeta^{(\nu)}(s+i\tau) - f(s)\right| < \epsilon\right\} > 0.$$

In [17] the author obtained Theorem C below. This theorem combines and improves Theorems A and B, and in fact provides a *simultaneous* universality theorem for the Riemann zeta-function $\zeta(s)$ and its derivatives $\zeta^{(j)}(s)$.

In this paper, subsets $K_1, \dots, K_n \subset \mathbb{C}$ are said to be *disjoint* if $K_j \cap K_{j'}$ is empty for any pair (j, j') with $j \neq j'$. As usual, let $\zeta^{(0)}(s)$ denote $\zeta(s)$.

Theorem C. *Let $m \in \mathbb{N}$. Let $\mathcal{K}_0, \mathcal{K}_1, \dots, \mathcal{K}_m$ be disjoint compact subsets of $\mathcal{D}$ such that $\mathbb{C}\backslash\mathcal{K}_j$ is connected for each $j = 0,1,\dots,m$. Let $f_0(s)$ be a nonvanishing continuous function on $\mathcal{K}_0$ which is holomorphic in the interior (if any) of $\mathcal{K}_0$. For each $j = 1,\dots,m$, let $f_j(s)$ be a continuous function on $\mathcal{K}_j$ which is holomorphic in the interior (if any) of $\mathcal{K}_j$. Then for any $\epsilon > 0$ we have*

$$\liminf_{T\to\infty}\frac{1}{T}\operatorname{meas}\left\{\tau \in [0,T]\colon \max_{0\le j\le m}\max_{s\in\mathcal{K}_j}\left|\zeta^{(j)}(s+i\tau) - f_j(s)\right| < \epsilon\right\} > 0.$$

For a prime q, let $\mathcal{X}_q$ denote the set of Dirichlet characters $\bmod q$. We have $\#\mathcal{X}_q = q - 1$. Let $L(s,\chi)$ denote the Dirichlet L-function associated to $\chi \in \mathcal{X}_q$. As an analogue of Theorem A, Bagchi [1] and Gonek [7] independently obtained the universality theorem for the family of Dirichlet L-functions $\{L(s,\chi)\colon \chi \in \mathcal{X}_q\}$ in the q-aspect (see Theorem D below). This theorem asserts, roughly, that any nonvanishing analytic function can be uniformly approximated by some Dirichlet L-functions $L(s,\chi) \bmod q$, where q is any sufficiently large prime. Notice that in Theorem A we consider the family of shifted functions $\{\zeta(s+i\tau)\colon \tau > 0\}$. For a related result, see [4], which deals with the case that the modulus of characters χ is the power q_0^n of a fixed prime q_0 with n sufficiently large. See also [16].

We now state the above theorem of Bagchi and Gonek in the modern form, which is shown in [16, Theorem 1] with $r = 1$ and $k_1 = 1$. Let $\mathbb{P}$ denote the set of prime numbers.

Theorem D. *Let $\mathcal{K}$ be a compact set in the strip $\mathcal{D}$ such that $\mathbb{C}\backslash\mathcal{K}$ is connected, and let $f(s)$ be a nonvanishing continuous function on $\mathcal{K}$ which is holomorphic in the interior (if any) of $\mathcal{K}$. Then for any $\epsilon > 0$ we have*

$$\liminf_{q\to\infty,\, q\in\mathbb{P}}\frac{1}{\#\mathcal{X}_q}\#\left\{\chi \in \mathcal{X}_q\colon \max_{s\in\mathcal{K}}|L(s,\chi) - f(s)| < \epsilon\right\} > 0.$$

The present paper has two purposes. The first one is to establish Theorem 1 below. This theorem is a generalization of Theorem D. Here, as usual, let $L^{(0)}(s,\chi)$ denote $L(s,\chi)$.

Theorem 1. *Let $m \in \mathbb{N}$. Let $\mathcal{K}_0, \mathcal{K}_1, \dots, \mathcal{K}_m$ be disjoint compact subsets of $\mathcal{D}$ such that $\mathbb{C}\backslash\mathcal{K}_j$ is connected for each $j = 0,1,\dots,m$. Let $f_0(s)$ be a nonvanishing continuous function on $\mathcal{K}_0$ which is holomorphic in the interior (if any) of $\mathcal{K}_0$. For each $j = 1,\dots,m$, let $f_j(s)$ be a continuous function on $\mathcal{K}_j$ which is holomorphic in the interior (if any) of $\mathcal{K}_j$. Then for any $\epsilon > 0$ we have*

$$\liminf_{q\to\infty,\, q\in\mathbb{P}}\frac{1}{\#\mathcal{X}_q}\#\left\{\chi \in \mathcal{X}_q\colon \max_{0\le j\le m}\max_{s\in\mathcal{K}_j}\left|L^{(j)}(s,\chi) - f_j(s)\right| < \epsilon\right\} > 0. \tag{1}$$

We now turn to the second main topic of this paper. For a meromorphic function $f(s)$ on the complex plane $\mathbb{C}$ and a complex number a, the zeros of the function $F(s) := f(s) - a$ are called the a-points of $f(s)$. An a-point ρ of $f(s)$ is said to be *simple* if $F'(\rho)(= f'(\rho)) \neq 0$.

In [23] Voronin also obtained the following two theorems on a-points of $\zeta(s)$ and its derivatives.

Theorem E. *Let $m \in \mathbb{N}$. Let Ω be a region in $\mathbb{C}^m$ of positive Jordan measure such that $(z_1,\dots,z_m) \in \Omega$ implies $1/2 < \operatorname{Re} z_j \le 1$ for any $j = 1,\dots,m$. Let $a_1,\dots,a_m \in \mathbb{C}\backslash\{0\}$ and $h > 0$. Then the number of solutions $(s_1,\dots,s_m)$ to the system of equations*

$$\begin{gathered}\zeta(s_1) = a_1,\\ \vdots\\ \zeta(s_m) = a_m\end{gathered}$$

such that

$$(s_1,\dots,s_m) \in \bigcup_{n=1}^{N}\{\Omega + in(h,\dots,h)\}$$

is greater than cN for all large $N \in \mathbb{N}$, where $c = c(\Omega, a_1, \ldots, a_m, h)$ is some positive constant.

Theorem F. *Let $m \in \mathbb{N}$. Let Ω be a region in $\mathbb{C}^{m+1}$ of positive Jordan measure such that $(z_0, z_1, \ldots, z_m) \in \Omega$ implies $1/2 < \mathrm{Re}\, z_j \leq 1$ for any $j = 0,1,\ldots,m$. Let $a_0 \in \mathbb{C}\backslash\{0\}$, $a_1, \ldots, a_m \in \mathbb{C}$ and $h > 0$. Then the number of solutions $(s_0, s_1, \ldots, s_m)$ to the system of equations*

$$\begin{aligned} \zeta(s_0) &= a_0, \\ \zeta'(s_1) &= a_1, \\ &\vdots \\ \zeta^{(m)}(s_m) &= a_m \end{aligned}$$

such that

$$(s_0, s_1, \ldots, s_m) \in \bigcup_{n=1}^{N} \{\Omega + in(h, h, \ldots, h)\}$$

is greater than cN for all large $N \in \mathbb{N}$, where $c = c(\Omega, a_0, a_1, \ldots, a_m, h)$ is some positive constant.

According to the Riemann hypothesis, the zeros of the Riemann zeta-function $\zeta(s)$ in the critical strip should be on the critical line. It is known that the Riemann zeta-function $\zeta(s)$ has infinitely many simple zeros on the critical line. This was first proved by Heath-Brown and Selberg independently (see [9]). Garunkštis and Steuding [6, Theorem 2] first proved that, for any complex number a, there are infinitely many simple a-points of $\zeta(s)$ with arbitrarily large imaginary part. Later, Gonek, Lester and Milinovich [8, Theorem 4] showed Theorem G below on simple a-points of $\zeta(s)$ by using Voronin's original universality theorem [24]. Further, in [17] the author also obtained a theorem which improves Theorems E, F and G. Notice that the a_j-points in Theorems E and F are not necessarily simple.

Theorem G. *Let $\sigma_1, \sigma_2 \in \mathbb{R}$ with $1/2 < \sigma_1 < \sigma_2 < 1$, and let $a \in \mathbb{C}\backslash\{0\}$. For $T > 0$, let $N(T; a, \sigma_1, \sigma_2)$ denote the number of simple a-points of $\zeta(s)$ in the rectangle $\{s \in \mathbb{C}\colon \sigma_1 < \mathrm{Re}\, s < \sigma_2,\ 0 < \mathrm{Im}\, s < T\}$. Then*

$$N(T; a, \sigma_1, \sigma_2) \gg T \quad \text{as } T \to \infty,$$

where the constant implied by $\gg$ may depend on a, σ_1 and σ_2.

The second purpose of the present paper is to investigate the existence and distribution of simple a-points for the family of Dirichlet L-functions $L(s,\chi)$ with $\chi \in \mathcal{X}_q$ and their derivatives. In fact, we establish Theorem 2 below. This theorem is an analogue of a theorem in [17] mentioned above. We note that Theorem 2 is obtained from Theorem 1.

For $s_0 \in \mathbb{C}$ and $r > 0$, let $B(s_0, r)$ denote the open disk $\{s \in \mathbb{C}\colon |s - s_0| < r\}$. A natural question is the following: Given an arbitrary complex number $a \neq 0$ and an arbitrary open set $B(s_0, r) \subset \mathcal{D}$, does there exist a Dirichlet L-function $L(s,\chi)$ which has a simple a-point in $B(s_0, r)$? Theorem 2 answers this question in the affirmative.

Theorem 2. *Let $m, m', M \in \mathbb{N}$. Suppose that $a_{0k} \in \mathbb{C}\backslash\{0\}$ for any $k = 1, \ldots, m'$ and that $a_{jk} \in \mathbb{C}$ for any $j = 1, \ldots, m$ and $k = 1, \ldots, m'$. Suppose that $s_{jk} \in \mathcal{D}$ for any $j = 0,1,\ldots,m$ and $k = 1,\ldots,m'$. Let $r > 0$. For a prime q, let $\mathcal{C}_q(m, m', M, \{a_{jk}\}, \{s_{jk}\}, r)$ denote the set of Dirichlet characters $\chi \in \mathcal{X}_q$ such that $L^{(j)}(s,\chi)$ has at least M simple a_{jk}-points in $B(s_{jk}, r)$ for any $j = 0,1,\ldots,m$ and $k = 1,\ldots,m'$. Then*

$$\liminf_{q \to \infty,\, q \in \mathbb{P}} \frac{1}{\#\mathcal{X}_q} \#\{\chi \in \mathcal{X}_q\colon \chi \in \mathcal{C}_q(m, m', M, \{a_{jk}\}, \{s_{jk}\}, r)\} > 0. \tag{2}$$

The next corollary is immediately obtained from Theorem 2, by recalling that $\#\mathcal{X}_q = q - 1$. This result generalizes Theorem G to the family of Dirichlet L-functions and their derivatives in the q-aspect.

Corollary 1. *Let ν be a nonnegative integer. Suppose that $a \in \mathbb{C}\backslash\{0\}$ if $\nu = 0$ and $a \in \mathbb{C}$ if $\nu \neq 0$. Let $s_0 \in \mathcal{D}$ and $r > 0$. For a prime q, let $\tilde{\mathcal{C}}_q(\nu, a, s_0, r)$ denote the set of Dirichlet characters $\chi \in \mathcal{X}_q$ such that $L^{(\nu)}(s,\chi)$ has at least one simple a-point in $B(s_0, r)$. Then*

$$\#\tilde{\mathcal{C}}_q(\nu, a, s_0, r) \gg q \quad \text{as } q \to \infty \text{ with } q \in \mathbb{P},$$

where the constant implied by $\gg$ may depend on ν, a, s_0 and r.

Notation 1. In this paper, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{P}$ denote the sets of all complex numbers, real numbers, positive integers, and prime numbers, respectively. For a subset S of $\mathbb{C}$, let $\overline{S}$ denote the closure of S and let ∂S denote the boundary of S. For two subsets S_1, S_2 of $\mathbb{C}$, let $\mathrm{dist}(S_1, S_2)$ denote $\inf\{|s_1 - s_2|\colon s_1 \in S_1, s_2 \in S_2\}$. Let $\emptyset$ denote the empty set.

2. Preliminary results

In the proof of Theorem 1, we will use some results concerning topology on the complex plane $\mathbb{C}$. Lemmas 1, 3 and 4 below are given and proved in [17]. However, since the paper [17] has not yet been published, we reproduce their proofs here for the sake of completeness.

Lemma 1. *Let $n \in \mathbb{N}$ with $n \geq 2$. Let $K_1, \ldots, K_n$ be disjoint compact subsets of $\mathbb{C}$ such that $\mathbb{C}\backslash K_j$ is connected for each $j = 1, \ldots, n$. Then $\mathbb{C}\backslash\left(\bigcup_{j=1}^{n} K_j\right)$ is connected.*

Proof. We prove the lemma by induction on n. For $n = 2$, the assertion is Corollary 2 of [3] (see also Remark 3.1 of [19]).

Let m be an integer with $m \geq 2$. Assume that the assertion holds for $n = m$. We shall prove the assertion for $n = m + 1$. Let $K_1, \ldots, K_m, K_{m+1}$ be disjoint compact subsets of $\mathbb{C}$ such that $\mathbb{C}\backslash K_j$ is connected for each $j = 1, \ldots, m, m + 1$. Then, by assumption and the definition of *disjointness* (given in Section 1), the set $\mathbb{C}\backslash\left(\bigcup_{j=1}^{m} K_j\right)$ is connected. The two sets $\bigcup_{j=1}^{m} K_j$ and K_{m+1} are disjoint. The set $\bigcup_{j=1}^{m} K_j$ is a compact subset of $\mathbb{C}$. Therefore, applying Corollary 2 of [3] (with $K = \bigcup_{j=1}^{m} K_j$ and $L = K_{m+1}$) again yields that $\mathbb{C}\backslash\left(\bigcup_{j=1}^{m+1} K_j\right) = \mathbb{C}\backslash\left(\left(\bigcup_{j=1}^{m} K_j\right) \cup K_{m+1}\right)$ is connected. Thus the assertion holds for $n = m + 1$. This completes the proof. □

The next result is the well-known Jordan curve theorem.

Lemma 2. *Let C be a Jordan curve (i.e., a simple closed curve) in the complex plane $\mathbb{C}$. Then its complement $\mathbb{C}\backslash C$ consists of exactly two connected components. One of these components is bounded and the other is unbounded, and the curve C is the boundary of each component.*

As in Section 1, let $\mathcal{D}$ denote the set $\{s \in \mathbb{C}\colon 1/2 < \operatorname{Re} s < 1\}$. The next lemma is proved by using a well-known argument (see e.g. [20]).

Lemma 3. *Let K and L be disjoint compact subsets of $\mathbb{C}$ such that $\mathbb{C}\backslash K$ is connected. Suppose that $K \subset \mathcal{D}$. Then there exist an integer $N \geq 1$ and rectifiable Jordan curves (actually, Jordan polygons) $C_1, C_2, \ldots, C_N$ in $\mathcal{D}$ such that $K \subset \bigcup_{j=1}^{N} U_j \subset \bigcup_{j=1}^{N} \overline{U_j} \subset \mathcal{D}$ and $K \cap U_j \neq \emptyset$ for all $j = 1, \ldots, N$, and such that L and the closures $\overline{U_1}, \overline{U_2}, \ldots, \overline{U_N}$ are disjoint. Here, each U_j denotes the bounded connected component of $\mathbb{C}\backslash C_j$ (see Lemma 2).*

Proof. Let B be an open disk centered at the origin with a sufficiently large radius such that $K \subset B$. By condition, L is contained in the open set $\mathbb{C}\backslash K$, and $\mathbb{C}\backslash K$ is connected (thus path-connected). Since K is bounded, the set $\mathbb{C}\backslash K$ is unbounded. Therefore, we can construct a path-connected compact set L' such that $L \subset L' \subset \mathbb{C}\backslash K$ and such that L' contains points outside B (that is, $L' \cap (\mathbb{C}\backslash B) \neq \emptyset$).

We put $\delta = \min\{\operatorname{dist}(K, L'), \operatorname{dist}(K, \partial B), \operatorname{dist}(K, \partial\mathcal{D})\} > 0$. Consider a grid of closed squares in $\mathbb{C}$ such that the diagonal length of each square is less than δ. Let K' be the union of all squares in this grid which intersect K. Since K is compact, K' is a finite union of squares. Since the distance from any point in K' to K is less than $\delta \leq \operatorname{dist}(K, L')$, we have $K' \cap L' = \emptyset$. Similarly, since that distance is less than $\delta \leq \operatorname{dist}(K, \partial B)$, we have $K' \subset B$. Also, since that distance is less than $\delta \leq \operatorname{dist}(K, \partial\mathcal{D})$, K' is contained in $\mathcal{D}$.

Let K'' be the union of K' and all bounded connected components (holes) of $\mathbb{C}\backslash K'$. Since L' is a connected set and $K' \cap L' = \emptyset$ as above, L' is entirely contained within a single connected component of $\mathbb{C}\backslash K'$. Since $K' \subset B$, all bounded connected components of $\mathbb{C}\backslash K'$ must lie entirely within B. Thus, since L' contains points outside B, L' is entirely contained within the unbounded connected component of $\mathbb{C}\backslash K'$. Hence L' is disjoint from all bounded connected components of $\mathbb{C}\backslash K'$, which implies $K'' \cap L' = \emptyset$. Therefore, since $L \subset L'$, we have $K'' \cap L = \emptyset$.

Any bounded connected component of $\mathbb{C}\backslash K'$ lies entirely within the convex hull of K' (otherwise, if a point in such a component were outside the convex hull, it could be connected to infinity by a ray without intersecting K', contradicting that the component is bounded). Since the open strip $\mathcal{D}$ is convex and $K' \subset \mathcal{D}$, that convex hull is completely contained in $\mathcal{D}$. Hence it follows that K'' is also contained in $\mathcal{D}$.

Since K'' is a compact set formed by a finite union of closed squares, it consists of a finite number of connected components, which are disjoint, say $K_1'', K_2'', \ldots, K_N''$. By the construction of K'', the complement $\mathbb{C}\backslash K_j''$ is connected for each j. By making sufficiently small local adjustments to the boundary of each K_j'' to eliminate self-intersections where squares meet only at their corners (if any), the boundary $\partial K_j''$ forms a simple closed polygonal curve, which is our desired Jordan polygon C_j. Letting U_j be the interior of K_j'', we have $K \subset \bigcup_{j=1}^{N} U_j \subset \bigcup_{j=1}^{N} \overline{U_j} \subset \mathcal{D}$. If there is any component U_j such that $K \cap U_j = \emptyset$, we can simply discard it. Since $K \subset \bigcup_{j=1}^{N} U_j$, removing those components disjoint from K still leaves K fully covered by the remaining components. By discarding such redundant components and relabeling them (letting N denote the new total number), we may ensure that $K \cap U_j \neq \emptyset$ for all $j = 1, \ldots, N$. We conclude that all C_j are in $\mathcal{D}$, $K \subset \bigcup_{j=1}^{N} U_j \subset \bigcup_{j=1}^{N} \overline{U_j} \subset \mathcal{D}$, $K \cap U_j \neq \emptyset$ for all j, and L and the closures $\overline{U_j}$ are disjoint. This completes the proof. □

Lemma 4. *Let $n \in \mathbb{N}$ with $n \geq 2$. Let $K_1, \dots, K_n$ be disjoint compact subsets of $\mathbb{C}$ such that $\mathbb{C}\setminus K_j$ is connected for each $j = 1, \dots, n$. Suppose that $K_j \subset \mathcal{D}$ for each $j = 1, \dots, n$. Then there exist integers $N_1, \dots, N_n \geq 1$ and Jordan polygons $C_{j,k}$ ($j = 1, \dots, n$; $k = 1, \dots, N_j$) in $\mathcal{D}$ such that for each $j = 1, \dots, n$, $K_j \subset \bigcup_{k=1}^{N_j} U_{j,k} \subset \bigcup_{k=1}^{N_j} \overline{U_{j,k}} \subset \mathcal{D}$ and $K_j \cap U_{j,k} \neq \emptyset$ for all $k = 1, \dots, N_j$, and such that all the closures $\overline{U_{j,k}}$ ($j = 1, \dots, n$; $k = 1, \dots, N_j$) are disjoint. Here, each $U_{j,k}$ denotes the bounded connected component of $\mathbb{C}\setminus C_{j,k}$ (see Lemma 2).*

Proof. We shall construct the required Jordan polygons iteratively for each $j = 1, \dots, n$ by applying Lemma 3.

First, we shall construct the Jordan polygons $C_{1,k}$ for K_1. We put $L_1 = \bigcup_{j=2}^{n} K_j$. Note that L_1 is a compact set disjoint from K_1. By condition, $\mathbb{C}\setminus K_1$ is connected and $K_1 \subset \mathcal{D}$. Therefore, by Lemma 3 (with $K = K_1$ and $L = L_1$), there exist an integer $N_1 \geq 1$ and Jordan polygons $C_{1,k}$ ($k = 1, \dots, N_1$) in $\mathcal{D}$ such that $K_1 \subset \bigcup_{k=1}^{N_1} U_{1,k} \subset \bigcup_{k=1}^{N_1} \overline{U_{1,k}} \subset \mathcal{D}$ and $K_1 \cap U_{1,k} \neq \emptyset$ for all $k = 1, \dots, N_1$, and such that the closures $\overline{U_{1,k}}$ ($k = 1, \dots, N_1$) and L_1 are disjoint. Further, since $K_2, \dots, K_n$ are disjoint and $L_1 = \bigcup_{j=2}^{n} K_j$, it follows that the closures $\overline{U_{1,k}}$ ($k = 1, \dots, N_1$) and the remaining sets $K_2, \dots, K_n$ are disjoint.

Next, assume that for some integer m with $1 \leq m < n$, there exist integers $N_1, \dots, N_m \geq 1$ and Jordan polygons $C_{j,k}$ ($j = 1, \dots, m$; $k = 1, \dots, N_j$) in $\mathcal{D}$ such that for each $j = 1, \dots, m$, $K_j \subset \bigcup_{k=1}^{N_j} U_{j,k} \subset \bigcup_{k=1}^{N_j} \overline{U_{j,k}} \subset \mathcal{D}$ and $K_j \cap U_{j,k} \neq \emptyset$ for all $k = 1, \dots, N_j$, and such that all the closures $\overline{U_{j,k}}$ ($j = 1, \dots, m$; $k = 1, \dots, N_j$) and the remaining sets $K_{m+1}, \dots, K_n$ are disjoint. We shall then proceed to construct the polygons $C_{m+1,k}$ for K_{m+1}.

Let $\widetilde{K}_j = \bigcup_{k=1}^{N_j} \overline{U_{j,k}}$ for each $j = 1, \dots, m$. We put

$$L_{m+1} = \left(\bigcup_{j=1}^{m} \widetilde{K}_j \right) \cup \left(\bigcup_{j=m+2}^{n} K_j \right). \tag{3}$$

Note that L_{m+1} is a compact set disjoint from K_{m+1}. By condition, $\mathbb{C}\setminus K_{m+1}$ is connected and $K_{m+1} \subset \mathcal{D}$. Therefore, by Lemma 3 (with $K = K_{m+1}$ and $L = L_{m+1}$), there exist an integer $N_{m+1} \geq 1$ and Jordan polygons $C_{m+1,k}$ ($k = 1, \dots, N_{m+1}$) in $\mathcal{D}$ such that $K_{m+1} \subset \bigcup_{k=1}^{N_{m+1}} U_{m+1,k} \subset \bigcup_{k=1}^{N_{m+1}} \overline{U_{m+1,k}} \subset \mathcal{D}$ and $K_{m+1} \cap U_{m+1,k} \neq \emptyset$ for all $k = 1, \dots, N_{m+1}$, and such that the closures $\overline{U_{m+1,k}}$ ($k = 1, \dots, N_{m+1}$) and L_{m+1} are disjoint. Further, by assumption and (3), it follows that the closures $\overline{U_{j,k}}$ ($j = 1, \dots, m+1$; $k = 1, \dots, N_j$) and the remaining sets $K_{m+2}, \dots, K_n$ are disjoint.

Repeating this iterative process for $m = 1, 2, \dots, n-1$ yields the desired Jordan polygons $C_{j,k}$. This completes the proof. □

The next result is a celebrated theorem of Mergelyan (see e.g. [5, p. 97, Theorem 1]).

Lemma 5. *Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C}\setminus K$ is connected. Let f be a continuous function on K which is holomorphic in the interior (if any) of K. Then, for any $\epsilon > 0$, there exists a polynomial P such that*

$$\max_{s \in K} |f(s) - P(s)| < \epsilon.$$

3. Proof of Theorem 1

We shall now prove Theorem 1. According to Lemma 4 (in which we take $n = m + 1$ and $K_1 = \mathcal{K}_0, \dots, K_n = \mathcal{K}_m$), there exist integers $N_1, \dots, N_m \geq 1$ and Jordan polygons C_{jk} ($j = 1, \dots, m$; $k = 1, \dots, N_j$) in $\mathcal{D}$ satisfying the following two conditions: (i) for each $j = 1, \dots, m$, we have $\mathcal{K}_j \subset \bigcup_{k=1}^{N_j} U_{jk} \subset \bigcup_{k=1}^{N_j} \overline{U_{jk}} \subset \mathcal{D}$ and $\mathcal{K}_j \cap U_{jk} \neq \emptyset$ for all $k = 1, \dots, N_j$, and (ii) $\mathcal{K}_0$ and the sets $\overline{U_{jk}}$ ($j = 1, \dots, m$; $k = 1, \dots, N_j$) are disjoint. Here, each U_{jk} denotes the bounded connected component of $\mathbb{C}\setminus C_{jk}$ (see Lemma 2).

Let $\epsilon > 0$ be arbitrary. By virtue of Lemma 5, we have polynomials $p_1(s), \dots, p_m(s)$ such that

$$\max_{s \in \mathcal{K}_j} \left| f_j(s) - p_j(s) \right| < \frac{\epsilon}{3} \quad \text{for each } j = 1, \dots, m. \tag{4}$$

Every $p_j(s)$ is defined and holomorphic on $\mathbb{C}$. Hence, for every $j = 1, \dots, m$, there exist holomorphic functions (polynomials) $p_{j,j-1}(s), p_{j,j-2}(s), \dots, p_{j,0}(s)$ on $\mathbb{C}$ such that

$$\frac{d}{ds} p_{j,j-1}(s) = p_j(s), \quad \frac{d}{ds} p_{j,j-2}(s) = p_{j,j-1}(s), \quad \cdots, \quad \frac{d}{ds} p_{j,0}(s) = p_{j,1}(s). \tag{5}$$

For each $j = 1, \dots, m$ and $k = 1, \dots, N_j$, we define the constant

$$b_{jk} := \max_{s \in \overline{U}_{jk}} \left| p_{j,0}(s) \right| \tag{6}$$

and the function

$$g_{jk}(s) \coloneqq p_{j,0}(s) + b_{jk} + 1, \quad s \in \mathbb{C}. \tag{7}$$

Then by (5),

$$g_{jk}^{(j)}(s) = p_j(s). \tag{8}$$

We put

$$K \coloneqq \mathcal{K}_0 \cup \left(\bigcup_{j=1}^{m} \bigcup_{k=1}^{N_j} \overline{U_{jk}} \right) \subset \mathcal{D}. \tag{9}$$

Noting that $\mathcal{K}_0$ and the $\overline{U_{jk}}$'s are disjoint, we define the function h on K by

$$h(s) := \begin{cases} f_0(s) & \text{for } s \in \mathcal{K}_0, \\ g_{jk}(s) & \text{for } s \in \overline{U_{jk}} \ (j = 1, \dots, m;\ k = 1, \dots, N_j). \end{cases}$$

By (7) and (6) we see that the function g_{jk} has no zeros on $\overline{U_{jk}}$. This and the condition on f_0 imply that

$$\text{the function } h \text{ has no zeros.} \tag{10}$$

Let us take a small positive real number ϵ_0 such that

$$\frac{j!}{2\pi} \mathrm{length}(C_{jk}) \epsilon_0 \max_{z \in C_{jk}, s \in \mathcal{K}_j \cap U_{jk}} \left| (z - s)^{-(j+1)} \right| < \frac{\epsilon}{3} \tag{11}$$

for any $j = 1, \dots, m$ and $k = 1, \dots, N_j$, where $\mathrm{length}(C_{jk})$ denotes the length of C_{jk}, and such that

$$\epsilon_0 < \frac{\epsilon}{3}. \tag{12}$$

According to Lemma 2 and the definition of U_{jk}, the set $\mathbb{C} \backslash \overline{U_{jk}}$ is the same as the unbounded connected component of $\mathbb{C} \backslash C_{jk}$ for any j, k. Thus, in particular,

$$\mathbb{C} \backslash \overline{U_{jk}} \text{ is connected for any } j, k. \tag{13}$$

By condition, $\mathbb{C} \backslash \mathcal{K}_0$ is connected. Recall that $\mathcal{K}_0$ and the $\overline{U_{jk}}$'s are disjoint compact subsets of $\mathbb{C}$. Therefore, (9), (13) and Lemma 1 imply that

$$\mathbb{C} \backslash K \text{ is connected.} \tag{14}$$

Using (14), (10) and Theorem D, we obtain the following approximation on K:

$$\liminf_{q \to \infty, q \in \mathbb{P}} \frac{1}{\# \mathcal{X}_q} \# \left\{ \chi \in \mathcal{X}_q \colon \max_{s \in K} |L(s, \chi) - h(s)| < \epsilon_0 \right\} > 0. \tag{15}$$

Let q be a large prime. Suppose that χ is an element of $\mathcal{X}_q$ satisfying

$$\max_{s \in K} |L(s, \chi) - h(s)| < \epsilon_0. \tag{16}$$

Then by definition (9),

$$\max_{s \in \mathcal{K}_0} |L(s, \chi) - f_0(s)| < \epsilon_0 \tag{17}$$

and

$$\max_{1 \le j \le m, 1 \le k \le N_j} \max_{s \in C_{jk}} \left| L(s, \chi) - g_{jk}(s) \right| < \epsilon_0, \tag{18}$$

since $C_{jk} = \partial U_{jk} \subset \overline{U_{jk}}$ by Lemma 2.

By (8), Cauchy's integral formula for derivatives, (18) (in which we replace s by z) and (11), we see that χ satisfies the following: for any $j = 1, \dots, m$ and $k = 1, \dots, N_j$ and uniformly for $s \in \mathcal{K}_j \cap U_{jk}$ ($\neq \emptyset$),

$$\begin{aligned}
&\left|L^{(j)}(s,\chi)-p_j(s)\right| = \left|L^{(j)}(s,\chi)-g_{jk}^{(j)}(s)\right| \\
&= \left|\frac{j!}{2\pi i}\oint_{C_{jk}} \frac{L(z,\chi)-g_{jk}(z)}{(z-s)^{j+1}}dz\right| \\
&\le \frac{j!}{2\pi}\operatorname{length}(C_{jk})\max_{z\in C_{jk}}\left|L(z,\chi)-g_{jk}(z)\right|_{z\in C_{jk},s\in\mathcal{K}_j\cap U_{jk}}\max\left|(z-s)^{-(j+1)}\right| \\
&< \frac{\epsilon}{3}.
\end{aligned} \tag{19}$$

Since $\mathcal{K}_j \subset \bigcup_{k=1}^{N_j} U_{jk}$ for any $j=1,\dots,m$, it follows from (4), (17), (19) and (12) that χ satisfies

$$\begin{aligned}
&\max_{0\le j\le m}\max_{s\in\mathcal{K}_j}\left|L^{(j)}(s,\chi)-f_j(s)\right| \\
&\le \max_{s\in\mathcal{K}_0}|L(s,\chi)-f_0(s)| + \max_{1\le j\le m}\max_{s\in\mathcal{K}_j}\left|L^{(j)}(s,\chi)-f_j(s)\right| \\
&\le \max_{s\in\mathcal{K}_0}|L(s,\chi)-f_0(s)| + \max_{1\le j\le m}\max_{s\in\mathcal{K}_j}\left|L^{(j)}(s,\chi)-p_j(s)\right| + \frac{\epsilon}{3} \\
&< \epsilon.
\end{aligned}$$

This and (16) imply that

$$\begin{aligned}
&\left\{\chi\in\mathcal{X}_q\colon \max_{0\le j\le m}\max_{s\in\mathcal{K}_j}\left|L^{(j)}(s,\chi)-f_j(s)\right|<\epsilon\right\} \\
&\supset \left\{\chi\in\mathcal{X}_q\colon \max_{s\in K}|L(s,\chi)-h(s)|<\epsilon_0\right\},
\end{aligned}$$

from which and (15) we obtain (1). Theorem 1 is proved.

4. Proof of Theorem 2

We shall now prove Theorem 2. Let us take complex numbers z_{jkl} ($j=0,1,\dots,m$; $k=1,\dots,m'$; $l=1,\dots,M$) and a small positive real number $\delta<\min_{1\le k\le m'}|a_{0k}|$ such that

$$\overline{B(z_{jkl},\delta)}\subset B(s_{jk},r) \quad \text{and} \quad \overline{B(z_{jkl},\delta)}\subset\mathcal{D} \quad \text{for any } j,k,l \tag{20}$$

and such that

$$\text{the sets } \overline{B(z_{jkl},\delta)} \text{ are disjoint.} \tag{21}$$

For each $j=0,1,\dots,m$, we put

$$\mathcal{K}_j := \bigcup_{k=1}^{m'}\bigcup_{l=1}^{M}\overline{B(z_{jkl},\delta)} \subset \mathcal{D} \tag{22}$$

and define the function f_j on $\mathcal{K}_j$ by

$$f_j(s) := s - z_{jkl} + a_{jk} \quad \text{for } s\in\overline{B(z_{jkl},\delta)} \tag{23}$$

in view of (21).

Since $\delta<\min_{1\le k\le m'}|a_{0k}|$, the function $f_0(s)$ has no zeros on $\mathcal{K}_0$. By (21), the sets $\mathcal{K}_j$ ($j=0,1,\dots,m$) are disjoint. Thus it follows from Theorem 1 that

$$\liminf_{q\to\infty,\,q\in\mathbb{P}}\frac{1}{\#\mathcal{X}_q}\#\left\{\chi\in\mathcal{X}_q\colon \max_{0\le j\le m}\max_{s\in\mathcal{K}_j}\left|L^{(j)}(s,\chi)-f_j(s)\right|<\delta\right\}>0. \tag{24}$$

Let q be a large prime. Suppose that χ is an element of $\mathcal{X}_q$ satisfying

$$\max_{0\le j\le m}\max_{s\in\mathcal{K}_j}\left|L^{(j)}(s,\chi)-f_j(s)\right|<\delta. \tag{25}$$

We fix any integers j,k,l with $0\le j\le m, 1\le k\le m', 1\le l\le M$. Then by (22), (25) and (23), for any $s\in\partial B(z_{jkl},\delta)$ we have

$$\begin{aligned}
&\left|\left(L^{(j)}(s,\chi)-a_{jk}\right)-\left(f_j(s)-a_{jk}\right)\right| = \left|L^{(j)}(s,\chi)-f_j(s)\right| \\
&< \delta = \left|s-z_{jkl}\right| = \left|f_j(s)-a_{jk}\right|.
\end{aligned}$$

The function $f_j(s) - a_{jk} = s - z_{jkl}$ has only one zero, counted with multiplicity, in $B(z_{jkl}, \delta)$. Hence by Rouché's theorem, the function $L^{(j)}(s, \chi) - a_{jk}$ has a zero of multiplicity one in $B(z_{jkl}, \delta)$, which implies that the function $L^{(j)}(s, \chi)$ has a simple a_{jk}-point in $B(z_{jkl}, \delta)$. Therefore by definition, (20) and (21), we have

$$\chi \in \mathcal{C}_q\big(m, m', M, \{a_{jk}\}, \{s_{jk}\}, r\big).$$

This, (25) and (24) yield (2). Theorem 2 is proved.

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