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| RESEARCH ARTICLE

Generating New Family of Continuous Distributions Using the Composition of Two Transformations

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| ABSTRACT
This paper introduces a novel family of probability distributions constructed by composing a Kavya-Manoharan (KM) generator with Exponentiated Bell- G family namely Kavya Manoharan Exponentiated Bell- G family. For this new composite family, we develop a comprehensive theoretical framework for the proposed family, including explicit forms of the cumulative distribution function, probability density function, survival and hazard functions, quantile function, linear representation, moments, incomplete moments and four different entropy measures are investigated to characterize the uncertainty of the proposed distribution. Maximum likelihood employed to estimate new family parameters and its finite-sample performance is evaluated through a simulation stud. The proposed family was applied to several real-world datasets to demonstrate its utility, utilizing the Rayleigh distribution as the baseline distribution. Goodness-of-fit results indicated that the proposed family of distributions provided a fit superior to that of many classical and modern distributions found in the literature. Thus, the proposed work offers a flexible framework for statistical modeling, capable of generating new distribution variants suited for analyzing real-world data.

| KEYWORDS
Kavya-Manoharan (KM) generator, Exponentiated Bell- G, uncertainty, analyzing, flexible.

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1. Introduction and Literature Review

Families of probability distributions constitute a cornerstone of mathematical statistics, offering an excellent approach to developing and extending probability distributions and enhancing their capacity to model diverse real-world data patterns. Constructing new families enhances the flexibility of classical distributions by improving their ability to represent varying degrees of skewness and kurtosis, as well as different shapes of probability density functions and hazard rates. The composition method is a significant technique for generating continuous statistical families; it relies on combining two families already established in the literature an approach employed by numerous researchers, such as: Topp-Leone Cauchy family of distributions by Atchadé, et al. (2023)., A new Topp-Leone Kumaraswamy Marshall-Olkin generated family of distributions by Atchadé, et al. (2024). A new power Topp–Leone generated family of distributions by Bantan, et al. (2019). A New Cosine Topp–Leone Exponentiated Half Logistic-G Family of distributions introduced by Chipepa, et al. (2026). The Topp-Leone Marshall-Olkin-G family of distributions proposed by Chipepa, et al. (2020), The Burr III-Topp-Leone-G family of distributions proposed by Chipepa, et al. (2021). DUS Topp–Leone-G family of distributions introduced by Ekemezie, et al. (2024), The Topp-Leone type II exponentiated half logistic-G family of distributions defined by Gabanakgosi, M., and Oluyede, B. (2023). The Marshall-Olkin-kumaraswamy-G family of distributions introduced by Handique, et al. (2017). A new generalized family of distributions based on combining Marshal-Olkin transformation with T-X family were proposed by Klakattawi, et al. (2022). Moakofi, T., Oluyede, B., and Gabanakgosi, M. (2022) introduced new family of continues distributions namely Topp-Leone Odd Burr III-G family of distributions. Nanga, et al (2023), proposed Cosine Topp–Leone family of distributions. Oluyede, B., and Moakofi, T. (2023) introduced gamma-Topp-Leone-type II-exponentiated half logistic-G family of distributions. Pu, et al (2023), defined new family by using compound method namely Ristić–Balakrishnan–Topp–Leone–Gompertz-G Family of distributions. The Marshall-Olkin-Type II-Topp-Leone-G family of distributions was introduced by, Rannona, et al. (2022). Watthanawisut, et al. (2022) proposed another new family were built by compound the beta and Topp-Leon families namely beta Topp-Leone generated family of distributions.

Gap of research: Although statistical families have demonstrated an ability to make traditional distributions more flexible and capable of modeling complex data, they remain limited in representing much of the data associated with the rapid developments of the modern world. Consequently, researchers are turning to approaches that combine existing families from the scientific literature to create new ones, thereby further enhancing the flexibility of traditional distributions.

Scientific contribution: The scientific contribution of this research paper lies in introducing a new probability family generated by combining existing statistical families, thereby leveraging the distinctive characteristics of each and enhancing the distributional flexibility of the resulting models. This construction enables the generation of a range of new distributions featuring more diverse shapes for their probability density and hazard rate functions, alongside the derivation of their mathematical and statistical properties and parameter estimation methods. Furthermore, the study assesses the added value of this combination by comparing the resulting model against the baseline distribution and the models generated from each constituent family individually, thereby determining whether the integration yields a genuine improvement in real-world data modeling.

2. Construction and Definition of KMEB-G Family

Kavya and Manoharan in (2021), introduced new transformation to generating a continues distributions with the following Cdf and Pdf respectively:

$$F(y,\xi) = \Theta\left(1 - e^{-G(y,\xi)}\right) \tag{1}$$

$$f(y,\xi) = \Theta\, g(y,\xi) e^{-G(y,\xi)} \tag{2}$$

Where $\Theta = \frac{e}{e-1}$ and $G(y,\xi), g(y,\xi)$ are the baseline cdf and pdf.

On the other hand Fayomi, et al. (2022) introduced Exponentiated Bell- G family which have the following Cdf and Pdf respectively:

$$G(y,\xi) = P\left(1 - e^{\left(-e^{\rho}\left[(H(y,\xi))^{\theta}\right]\right)}\right) \quad , \tag{3}$$

$$g(y,\xi) = P\lambda\theta h(y,\xi)(H(y,\xi))^{\theta-1} e^{\lambda(1-(H(y,\xi))^{\theta})} e^{\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)} \tag{4}$$

Where $P = \frac{1}{1-e^{\left(1-e^{\lambda}\right)}}$ $y > 0, \rho, \theta > 0$ and $H(y,\xi), H(y,\xi)$are the baseline cdf and pdf.

Now, by composing equations (3,4) with (1,2), we obtain the new family, which comprises the following cumulative and density distribution functions:

$$F(y,\phi) = \Theta\left(1 - e^{-P\left(1-e^{\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)}\right)}\right) \tag{5}$$

$$f(x,\phi) = \Theta P\lambda\theta h(x,\xi)\big(H(y,\xi)\big)^{\theta-1} e^{\lambda\left(1-(H(y,\xi))^{\theta}\right)} e^{\left(-e^{\lambda}\left[1-e^{-\lambda\left(1-(H(y,\xi))^{\theta}\right)}\right]\right)} * e^{-P\left(1-e^{\left(-e^{\lambda}\left[1-e^{-\lambda\left(1-(H(y,\xi))^{\theta}\right)}\right]\right)}\right)} \tag{6}$$

3. Statistical Properties for KMEB-G Family

3.1 Linear Representation for Probability Density Function

The linear expansion of the probability density function (PDF) is significant because it transforms a complex density function into a simpler one, thereby facilitating the study of the new distribution's mathematical properties. Its importance is particularly evident in simplifying the derivation of moments, incomplete moments, the moment-generating function, the characteristic function, entropy, and order statistics, as opposed to dealing directly with the complex original form. So the linear representation for KMEB-G family probability density function will be found by used the exponential and binomial Series expansions as followed:

$$e^{-az} = \sum_{m=0}^{\infty} \frac{(-a)^m}{m!}(z)^m, \qquad (1-z)^a = \sum_{m=0}^{\infty} \binom{a}{m}(-1)^m (z)^m, \;\; |z| < 1$$

Now from eq.(6)

$$e^{-P\left(1-e^{\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)}\right)} = \sum_{i=0}^{\infty} \frac{(-P)^i}{i!}\left(1 - e^{\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)}\right)^i$$

$$\left(1 - e^{\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)}\right)^i = \sum_{n=0}^{\infty}\binom{i}{n}(-1)^n e^{n\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)}$$

So that

$$f(y,\phi) = \Theta\, P\lambda\theta\, h(y,\xi)\big(H(y,\xi)\big)^{\theta-1} e^{\lambda\left(1-(H(y,\xi))^{\theta}\right)} * \sum_{n,i=0}^{\infty}\binom{i}{n}\frac{(-1)^n(-P)^i}{i!} e^{(n+1)\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)}$$

Again by the same way

$$e^{(n+1)\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)} = \sum_{d=0}^{\infty} \frac{(n+1)^d}{d!}\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)^i$$

$$f(y,\phi) = \Theta\, P\lambda\theta\, \left(-e^{\lambda}\right)^i h(y,\xi)\left(H(y,\xi)\right)^{\theta-1} e^{\lambda\left(1-(H(y,\xi))^{\theta}\right)} * \sum_{n,i,d=0}^{\infty} \binom{i}{n} \frac{(-1)^n (n+1)^d (-P)^i}{d!\, i!}\left(1-e^{-\lambda(H(y,\xi)^{\theta}}\right)^i$$

Now by used generalized binomial theorem we have

$$\left(1-e^{-\lambda(H(y,\xi)^{\theta}}\right)^i = \sum_{s=0}^{\infty} \binom{i}{s} (-1)^s e^{-s\lambda(H(y,\xi)^{\theta}}$$

That is

$$f(x,\phi) = \Theta\, P\lambda\theta\, \left(-e^{\lambda}\right)^i h(y,\xi)\left(H(y,\xi)\right)^{\theta-1} e^{\lambda\left(1-(H(y,\xi))^{\theta}\right)} * \sum_{n,i,d,s=0}^{\infty} \binom{i}{n}\binom{i}{s} \frac{(-1)^{n+s}(n+1)^d (-P)^i}{d!\, i!} e^{-s\lambda(H(y,\xi))^{\theta}}$$

Also by used exponential expansion we have $e^{\lambda\left(1-(H(y,\xi))^{\theta}\right)} = \sum_{z=0}^{\infty} \frac{\lambda^z}{z!}\left(1-\left(H(y,\xi)\right)^{\theta}\right)^z$

Then $f(y,\phi) = \Theta\, P\lambda\theta\, \left(-e^{\lambda}\right)^i h(y,\xi)\left(H(y,\xi)\right)^{\theta-1} *$

$$\sum_{n,i,d,s,z=0}^{\infty} \binom{i}{n}\binom{i}{s} \frac{(-1)^{n+s}(n+1)^d (-P)^i}{d!\, i!} \frac{\lambda^z}{z!}\left(1-\left(H(y,\xi)\right)^{\theta}\right)^z e^{-s\lambda(H(x,\xi)^{\theta}}$$

By the same way $\left(1-\left(H(y,\xi)\right)^{\theta}\right)^z = \sum_{m=0}^{\infty}\binom{z}{m}(-1)^m (H(y,\xi))^{m\theta}$

Then $f(y,\phi) = \Theta\, P\lambda\theta\, \left(-e^{\lambda}\right)^i \left[\sum_{n,i,d,s,z,m=0}^{\infty}\binom{z}{m}\binom{i}{n}\binom{i}{s}\frac{(-1)^{n+s+m}(n+1)^d(-P)^i}{d!i!}\frac{\lambda^z}{z!}\right] * h(y,\xi)(H(y,\xi))^{(m+1)\theta} e^{-s\lambda(H(y,\xi)^{\theta}}$

By $e^{-s\lambda(H(x,\xi)^{\theta}} = \sum_{k=0}^{\infty}\frac{(-s\lambda)^k}{k!}(H(y,\xi))^{k\theta}$

$$f(y,\phi) = \Theta\, P\lambda\theta \sum_{n,i,d,s,z,m,k=0}^{\infty} \binom{z}{m}\binom{i}{n}\binom{i}{s}\left(-e^{\lambda}\right)^i \frac{(-1)^{n+s+m}(n+1)^d(-s\lambda)^k(-P)^i}{k!\, d!\, i!}\frac{\lambda^z}{z!} * h(y,\xi)(H(y,\xi))^{(m+1)\theta+k\theta}$$

Now let

$$\Psi_{n,i,d,s,z,m,k} = \Theta\, P\lambda\theta \sum_{n,i,d,s,z,m,k=0}^{\infty} \binom{z}{m}\binom{i}{n}\binom{i}{s}\left(-e^{\lambda}\right)^i \frac{(-1)^{n+s+m}(n+1)^d(-s\lambda)^k(-P)^i}{k!\, d!\, i!}\frac{\lambda^z}{z!}$$

Then

$$f(y,\phi) = \Psi_{n,i,d,s,z,m,k} h(y,\xi)(H(y,\xi))^{(m+1)\theta+k\theta} \qquad (7)$$

And

$$\left(f(y,\phi)\right)^{\vartheta} = \left(\Psi_{n,i,d,s,z,m,k}\right)^{\vartheta}(h(y,\xi))^{\vartheta}(H(y,\xi))^{\vartheta(m+1)\theta+k\theta} \qquad (8)$$

3.2 Quantile function for KMEB-G family

A quantile function provides the value below which a given percentage of data falls. Given a cumulative distribution function, so to derive quantile function for KMEB-G family we inverted Eq. (3) as follow:

$$q = \Theta\left(1-e^{-G(y,\phi)}\right)$$

$$\frac{q}{\Theta} = 1-e^{-G(y,\phi)}$$

$$e^{-G(y,\phi)} = 1-\frac{q}{\Theta}$$

$$G(y,\phi) = -ln\left\{1-\frac{q}{\Theta}\right\}$$

$$y = G^{-1}\left(-ln\left\{1-\frac{q}{\Theta}\right\},\phi\right)$$

Eq.(7) refer to quantile function for KMEB –G family, median for new family defined as follow:

Median = $G^{-1}\left(-ln\left\{1-\frac{0.5}{\Theta}\right\},\phi\right)$

3.3 Moments for KMEB-G family

Moments are among the most important mathematical tools, as they can be used to analyze and describe mathematical functions and various real-world datasets. Given this significance, they have become indispensable across diverse fields, offering deep insight and high precision for making predictions in both theoretical and applied research. So that the moments for KMEB-G family calculated as follows:

If Y is continues R.V then the moments have the following form

$\mu_r = \int_{-\infty}^{\infty} y^r g(y,\varphi)\, dy$, where $g(y,\varphi)$ is the Pdf for Y

Now by Using the equation (7) , we have

$$\mu_r = \Psi_{n,i,d,s,z,m,k} \int_0^{\infty} y^r h(y,\xi)(H(y,\xi))^{(m+1)\theta+k\theta}\, dy \quad (10)$$

3.4 Incomplete Moments for KMEB-G family

Incomplete moments are among the important statistical properties of probability distributions, as they are used to study the behavior of a random variable within a specific portion of the distribution's domain. They also play a significant role in deriving various important measures, such as mean deviation and Lorenz and Bonferroni curves. So that the incomplete moments for KMEB-G family calculated as follows:

If Y is continues R.V then the incomplete moments have the following form

$\mu_t = \int_{-\infty}^{t} y\, g(y,\varphi)\, dy$, where $g(y,\varphi)$ is the Pdf for Y

Now by Using the equation (7) , we have

$$\mu_r = \Psi_{n,i,d,s,z,m,k} \int_0^{t} y h(y,\xi)(H(y,\xi))^{(m+1)\theta+k\theta}\, dy \quad (11)$$

3.5 Moment Generating Function for KMEB-G family

The moment generating function for KMEB –G family say $M_Y(t)$ is obtained from the following form

$$M_Y(t) = \sum_{d=0}^{\infty}\frac{t^d}{d!}\mu_r$$

Hence the moment generating function for our family is given by

$$M_X(t) = \sum_{d=0}^{\infty}\frac{t^d}{d!}\Psi_{n,i,d,s,z,m,k} \int_0^{t} y h(y,\xi)(H(y,\xi))^{(m+1)\theta+k\theta}\, dy \quad (12)$$

3.6 Order Statistics for KMEB-G family

Let $X_1, X_2, X_3, \dots, X_n$ be a random variables of size n from KMEB –G family then PDF of the d^{th} order statistics respectively given by the following form:

$$g_d(y) = \frac{l!}{(d-1)!\,(l-d)!}\sum_{k=0}^{l-d} g(y)\; G^{k+d-1}(y)$$

So for KMEB–G family and by using eqs.(5) and (7) we get that

$$g_d(y) = \frac{l!}{(d-1)!\;(l-d)!}\sum_{k=0}^{l-d}\Psi_{n,i,d,s,z,m,k} h(y,\xi)\big(H(y,\xi)\big)^{(m+1)\theta+k\theta} *$$

$$\left[\left(1-e^{-P\left(1-e^{\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)}\right)}\right)\right]^{k+d-1} \quad (13)$$

3.7 Reverse Order Statistics for KMEB-G family

The reversed order statistics can be used when $x_1, x_2, x_3, \dots, x_n$are arranged in the decreasing order; the PDF of $X_{d(re):l}$ is given by:

$$g_{d(re):l}(y,\varphi) = \binom{l}{d}\sum_{h=0}^{d-1}(-1)^h\binom{r-1}{l} g(y,\varphi)(G(y,\varphi))^{l-d+h}$$

So for KMEB–G family and by using eqs.(5) and (7) we get that

$$g_{d(re):l}(y,\varphi) = \binom{l}{d}\sum_{h=0}^{d-1}(-1)^h\binom{r-1}{l}\Psi_{n,i,d,s,z,m,k} h(y,\xi)\big(H(y,\xi)\big)^{(m+1)\theta+k\theta} * \left(\left(1-e^{-P\left(1-e^{\left(-e^{\lambda}\left[1-e^{-\lambda(H(y,\xi)^{\theta}}\right]\right)}\right)}\right)\right)^{l-d+h} \tag{14}$$

3.8 KMEB-G family Inequality Measures

In this section two inequality measures will be introduced which are Lorenz and Bonferroni curves that utilized in various fields such as insurance, econometrics, and demography, among others, to analyze measures of inequality such as income and poverty.

3.8.1 Lorenz curve

The Lorenz curve is one of the most important measures based on incomplete moments; it is used to quantify inequality in the distribution of a random variable. It helps illustrate how values are concentrated within the distribution feature that has led to its widespread use in economic studies—and can also be employed in the analysis of lifetime and reliability distributions to characterize distribution properties more deeply. This curve can be defined as:

$L_{F(x)} = \frac{1}{\mu}\int_{-\infty}^{x} y g(y,\varphi)\,dy$ .Where μ the first moment for y. Now for KMEB-G family Lorenz curve given as:

$$L_{F(y)} = \frac{1}{\mu}\ \Psi_{n,i,d,s,z,m,k}\int_{-\infty}^{x} y h(y,\xi)(H(y,\xi))^{(m+1)\theta+k\theta}\,dy \tag{15}$$

3.8.2 Bonferroni curve

Bonferroni curve is defined as $B_{F(x)} = \frac{L_{F(x)}}{F(x)}$ where$F(x)$ is the Cdf where defined in eq.(3) with respect to x so that Bonferroni curve for KMEB –G family is given as:

$$B_{F(x)} = \frac{1}{F(x)}\frac{1}{\mu}\ \Psi_{n,i,d,s,z,m,k}\int_{-\infty}^{x} y h(y,\xi)(H(y,\xi))^{(m+1)\theta+k\theta}\,dy \tag{16}$$

3.9 Information and Uncertainty Measures

The randomness and uncertainty associated with any random variable are measured by the entropy function. This function is used in various applications due to its importance in demonstrating the above in different fields, such as medicine, life sciences, insurance, engineering, and various economic fields. It is used as a scientifically valuable tool for the quantity of information in analyzing systems and measuring irregularity within them.

3.9.1 Information Generating Function

The Information Generating Function (IGF) is highly significant as it provides a unified tool for studying uncertainty and information content within a probability distribution. Furthermore, its values and derivatives allow it to be linked to various important information measures, such as Rényi entropy. For continuous distributions, it can be defined by the following formula:

$$IGF = \int_0^{\infty} g^{\tau}(x), \quad \tau \neq 0\,, \tau > 1$$

So that for KMEB-G family and by used eq.(8) the information generating function is given by the form:

$$IGF = \left(\Psi_{n,i,d,s,z,m,k}\right)^{\tau}\int_0^{\infty}(h(y,\xi))^{\tau}(H(y,\xi))^{\tau\left((m+1)\theta+k\theta\right)}, \quad \tau \neq 0\,, \tau > 1 \tag{17}$$

3.9.2 Rényi Entropy

Rényi entropy Rényi entropy is one of the most important measures of uncertainty and information; it is characterized by a parameter known as the entropy index, which allows for controlling the degree of sensitivity to different parts of the probability distribution. For a continuous random variable, it is defined as:

$R_E(\upsilon) = \frac{1}{1-\upsilon}log\left[\int_0^{\infty} g^{\upsilon}(y)dy\right]. \qquad \upsilon > 0\,, \upsilon \neq 1.$

So by using eq.(8), t$R_E(\upsilon)$ for KMEB –G family is given by:

$$R_E(\upsilon) = \frac{1}{1-\upsilon}log\left[\left(\Psi_{n,i,d,s,z,m,k}\right)^{\upsilon}\int_{-\infty}^{\infty}(h(y,\xi))^{\upsilon}(H(y,\xi))^{\upsilon\left((m+1)\theta+k\theta\right)}dy\right]. \tag{18}$$

Where$\upsilon > 0\,, \upsilon \neq 1$.

3.9.3 Arimoto Entropy

Arimoto entropy (A_E) compute by the following form:

$$A_E = \frac{\upsilon}{1-\upsilon}\left(\left[\int_0^\infty g^\upsilon(x)dx\right]^{\frac{1}{\upsilon}} - 1\right) \quad ,\upsilon > 0\,,\ \upsilon \neq 1.$$

Now by using eq. (8) getting that Arimoto entropy (A_E) for Sine Kavya Manoharan –G family is given by:

$$A_E = \frac{\upsilon}{1-\upsilon}\left(\left[\left(\Psi_{n,i,d,s,z,m,k}\right)^\upsilon \int_0^\infty \left(h(y,\xi)\right)^\upsilon \left(H(y,\xi)\right)^{\upsilon\left((m+1)\theta+k\theta\right)} dy\right]^{\frac{1}{\upsilon}} - 1\right) \tag{19}$$

Where , $\upsilon > 0\,,\ \upsilon \neq 1$.

3.9.4 Tsallis Entropy

Tsallis Entropy (TS_τ) can be computed from the following form:

$$T_E = \frac{1}{\upsilon-1}\left(1 - \int_0^\infty g^\upsilon(x)dx\right),\ \upsilon > 0\,,\ \upsilon \neq 1.$$

Now by using eq. (8) getting that T_E for KMEB –G family is given by:

$$TS_\tau = \frac{1}{\upsilon-1}\left(1 - \left[\left(\Psi_{n,i,d,s,z,m,k}\right)^\upsilon \int_0^\infty \left(h(y,\xi)\right)^\upsilon \left(H(y,\xi)\right)^{\upsilon\left((m+1)\theta+k\theta\right)} dy\right]\right) \tag{20}$$

Where , $\upsilon > 0\,,\ \upsilon \neq 1$.

4. Maximum likelihood Estimation Method

Let ($X_1, X_2, \dots, X_n$) are continues identically independent distribution (iid) random variables with observations ($x_1, x_2, \dots, x_n$) from the density function $f(y,\xi)$ where ξ is the vector of unknown parameters such that then the likelihood function L(ξ) is:

$$L(\xi) = \prod_{i=1}^n f(y_i,\xi)$$

So that for KMEB-G family likelihood function is given by:

$$L(\xi) = \prod_{i=1}^n P\Theta\lambda\theta h(y_i,\xi)\left(H(y,\xi)\right)^{\theta-1} e^{\lambda\left(1-\left(H(y_i,\xi)\right)^\theta\right)} e^{\left(-e^\lambda\left[1-e^{-\lambda\left(1-\left(H(y_i,\xi)\right)^\theta\right)}\right]\right)} * e^{-P\left(1-e^{\left(-e^\lambda\left[1-e^{-\lambda\left(1-\left(H(y_i,\xi)\right)^\theta\right)}\right]\right)}\right)}$$

$$L(\lambda,\theta,\xi) = n\log P + n\log\Theta + n\log\lambda + n\log\theta + \sum_{i=}^n log\, h(y_i,\xi) + (\theta-1)\sum_{i=1}^n log(H(y_i,\xi)) + \lambda\sum_{i=1}^n \left(1 - \left(H(y_i,\xi)\right)^\theta\right) +$$
$$\sum_{i=1}^n \left(-e^\lambda\left[1 - e^{-\lambda\left(1-\left(H(y_i,\xi)\right)^\theta\right)}\right]\right) - P\sum_{i=1}^n \left(1 - e^{\left(-e^\lambda\left[1-e^{-\lambda\left(1-\left(H(y_i,\xi)\right)^\theta\right)}\right]\right)}\right) \tag{21}$$

The next is to find the first partial derivative for $L(\lambda,\theta,\xi)$ then equal it to zero.

5. Kavya Manoharan Exponentiated Bell Rayleigh Distribution

The cumulative distribution function and probability density function for KMEBRay distribution can be developed by insert the following equations in eqs.(3) , (4) as follow..

$$H(y,b) = 1 - e^{-\frac{y^2}{2b^2}}; \qquad y \geq 0\,, b > 0$$

$$h(y;b) = \frac{y}{b^2}\, e^{-\frac{y^2}{2b^2}}; \qquad y \geq 0\,, b > 0$$

$$G(y,\xi) = P\left(1 - e^{\left(-e^\lambda\left[\left(1-e^{-\frac{y^2}{2b^2}}\right)^\theta\right]\right)}\right) \quad , \tag{23}$$

$$g(y,\xi) = P\lambda\theta h(x,\xi)\left(1-e^{-\frac{y^2}{2b^2}}\right)^{\theta-1} e^{\lambda\left(1-\left(1-e^{-\frac{y^2}{2b^2}}\right)^{\theta}\right)} e^{\left(-e^{\lambda}\left[1-e^{-\lambda\left(1-e^{-\frac{y^2}{2b^2}}\right)^{\theta}}\right]\right)} \tag{24}$$

Where $y \geq 0\,, b\,, \lambda, \theta > 0$

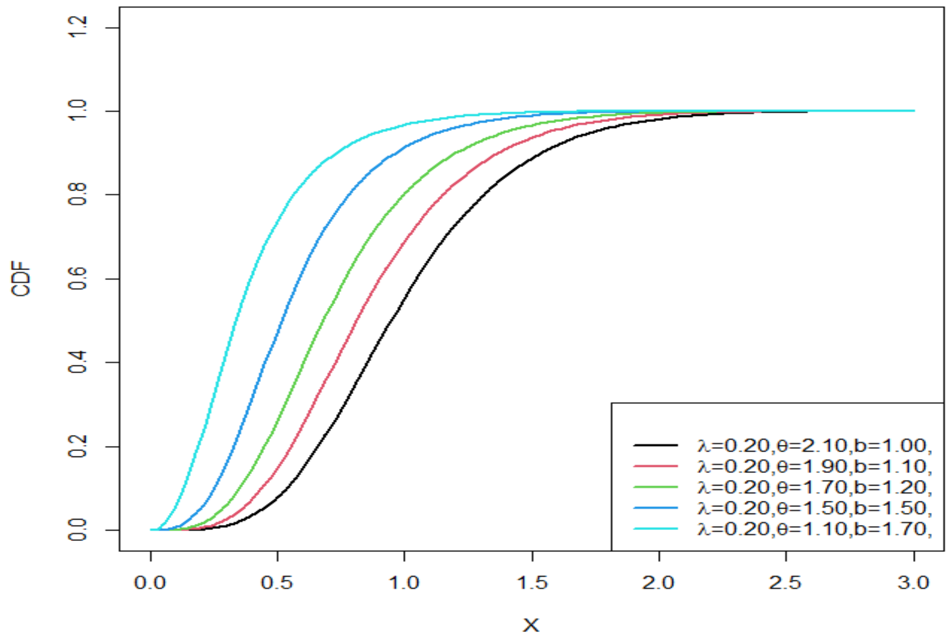

Figure 1. cdf shape for KMEBRay distribution

Figure 1. Explain the behavior of the cumulative distribution function for KMEBRay distribution across various sets of parameter values. All curves exhibit the behavior expected of a valid CDF: starting near zero, increasing monotonically with the variable, and gradually approaching one. The figure also demonstrates that varying the parameter values clearly influences the rate of probability accumulation and the curve's position; some parameter sets result in faster probability accumulation at lower variable values, whereas others require higher values to reach the same cumulative level. This variation reflects the distribution parameters' ability to govern its behavior, thereby supporting its flexibility in representing diverse data patterns.

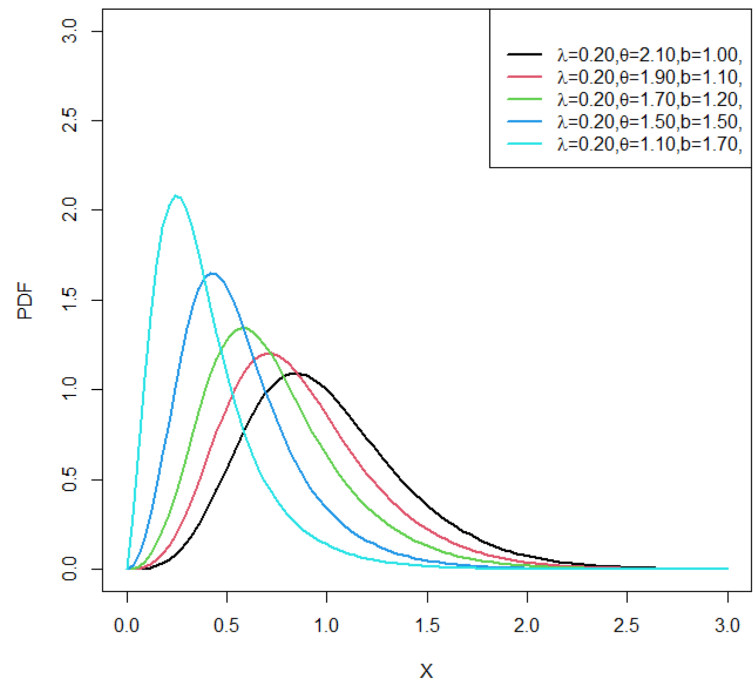

Figure 2. Pdf shape for KMEBRay distribution

Figure 2. Explain the behavior of the probability density function for KMEBRay distribution across various sets of parameter values. It is evident that the density function exhibits a unimodal and positively skewed (right-skewed) shape in all cases shown, with distinct variations in the peak's location and height, as well as the curve's spread, depending on the parameter values.

5.1 Quantile function for KMEBRay Distribution

The quantile function is one of the most important statistical properties of probability distributions; it plays a fundamental role in generating random variables via the inverse transform method, making it particularly significant in simulation studies and the comparison of parameter estimation methods. Furthermore, quantile values contribute to understanding the location, dispersion, and asymmetry of the data, thereby providing a more comprehensive description of the proposed distribution's behavior, so to derive quantile function for KMEBRay family we inverted Eq. (23) as follow:

$$q = P\left(1 - e^{\left(-e^{\lambda}\left[\left(1-e^{-\frac{y^2}{2b^2}}\right)^{\theta}\right]\right)}\right)$$

$$e^{\left(-e^{\lambda}\left[\left(1-e^{-\frac{y^2}{2b^2}}\right)^{\theta}\right]\right)} = \left(1-\frac{q}{P}\right)$$

$$e^{\lambda}\left[\left(1-e^{-\frac{y^2}{2b^2}}\right)^{\theta}\right] = -\log\left(1-\frac{q}{P}\right)$$

$$\left(1-e^{-\frac{y^2}{2b^2}}\right)^{\theta} = \frac{1}{\lambda}\log\left[-\,log\left(1-\frac{q}{P}\right)\right]$$

$$1-\sqrt[\theta]{\frac{1}{\lambda}log\left[-\,log\left(1-\frac{q}{P}\right)\right]} = e^{-\frac{y^2}{2b^2}}$$

$$-2b^2\left(log\left(1-\sqrt[\theta]{\frac{1}{\lambda}log\left[-\,log\left(1-\frac{q}{P}\right)\right]}\right)\right) = y^2$$

$$y = \sqrt{-2b^2 \left(log \left(1 - \sqrt[\theta]{\frac{1}{\lambda} log \left[- log \left(1 - \frac{q}{P} \right) \right]} \right) \right)} \quad (24)$$

The following table presents the numerical values of the quantile function, with all distribution parameters held constant while the probability level q varies within the range $0 < q < 1$. The results demonstrate that as q decreases, the corresponding quantile values decrease monotonically, reflecting a gradual transition from the upper to the lower levels of the distribution. This behavior aligns with the theoretical properties of the quantile function, wherein a lower probability level corresponds to an equal or smaller quantile value. This consistent pattern provides numerical support for the validity and behavior of the derived quantile function for the proposed distribution.

Table 1. set of parameters value

Set	λ	θ	b
Set 1	0.5	0.7	0.4
Set 2	0.9	1.2	0.6
Set 3	1.4	1.7	1.1
Set 4	1.9	2.1	1.3

Table 2. numerical behavior for quantile function

q	**Set 1**	**Set 2**	**Set 3**	**Set 4**
0.1	0.15555893	0.27345763	0.26057543	0.25760572
0.2	0.26816600	0.38056955	0.33078433	0.31265864
0.3	0.37879329	0.47187604	0.38671135	0.35471637
0.4	0.49569467	0.56044086	0.43847972	0.39235971
0.5	0.62565203	0.65335537	0.49078462	0.42919800
0.6	0.77748227	0.75774078	0.54767733	0.46794414
0.7	0.96600330	0.88447442	0.61485059	0.51196033
0.8	1.22234723	1.05636888	0.70409782	0.56764754
0.9	1.63869180	1.34421000	0.85400306	0.6547251

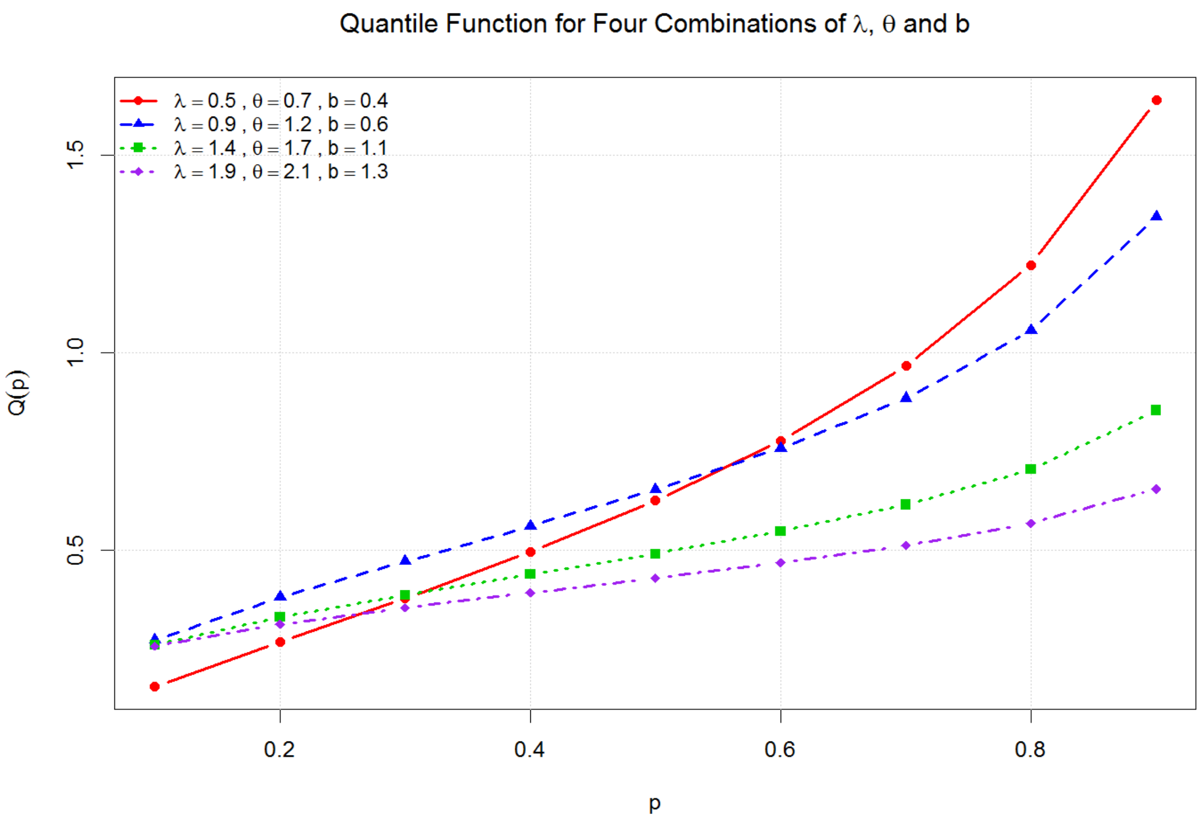

Figure 3. Quantile function for different Parameters set.

Figure 3. illustrates the quantile function values listed in the table across various probability levels and for four distinct sets of parameters for the proposed distribution; the plot demonstrates the consistent variation in quantile function values as q increases, thereby providing a visual representation of the function's behavior and how it is influenced by changes in the distribution parameters.

5.2 Information and Uncertainty Measures for KMEBRay distribution

We have previously presented the entropy and information measures for the new family; in this section, we present an application of these measures to a proposed distribution constructed using the family under study. The aim is to examine its informational behavior and illustrate how these measures are affected by changes in the entropy index and the distribution parameters.

Table 3. Numerical Values for Different Values of the Entropy

q	IGF	Renyi	Tsallis	Arimoto
1.1	1.08677256	-0.8321235	-0.8677256	-0.8644065
1.2	1.21031789	-0.9544152	-1.0515895	-1.0345145
1.3	1.38080604	-1.075558	-1.2693535	-1.2208021
1.4	1.61381014	-1.1964948	-1.5345253	-1.4264499
1.5	1.93311838	-1.3182689	-1.8662368	-1.6554345
1.6	2.37564653	-1.442116	-2.2927442	-1.9129842
1.7	3.00031547	-1.5695963	-2.8575935	-2.2062959
1.8	3.90493771	-1.7028023	-3.6311721	-2.5457554

Table 3. Numerical Values for Different Values of the Entropy

q	IGF	Renyi	Tsallis	Arimoto
1.9	5.26058323	-1.8447132	-4.7339814	-2.947166
2	7.38823162	-1.9998884	-6.3882316	-3.4362603

Table 3. presents a numerical study of entropy measures, which are crucial for elucidating the informational characteristics of the proposed distribution; theoretical formulas for these measures do not directly reveal how the level of uncertainty changes with variations in distribution parameters and the entropy index. The table shows that the three entropy measures: Rényi, Tsallis, and Arimoto, decrease as the entropy index increases, while the distribution parameters remain constant. Furthermore, the table illustrates the concentration and dispersion of the new distribution's probability density function based on the information-generating function. Consequently, these results provide a quantitative description that complements the density function and traditional distribution properties, aiding in the understanding of its informational behavior across various parameter settings.

6. Application

Practical application constitutes a crucial aspect of studying any new probability distribution; the significance of a distribution lies not merely in deriving its mathematical and statistical properties, but also in verifying its actual capacity to represent real-world data. Accordingly, this section aims to evaluate the practical performance of the proposed distribution and demonstrate its flexibility in fitting data, in comparison with a set of baseline and competing distributions such as Beta-Rayleigh Distribution, Kumaraswamy-Rayleigh distribution, Exponentiated Generalized Rayleigh distribution ,Weibull-Rayleigh distribution , Gompertz-Rayleigh distribution , Marshall–Olkin Rayleigh distribution and Rayleigh Distribution. This comparison is particularly significant in determining whether the distribution's new formulation has effectively enhanced its ability to characterize data features that baseline distributions may fail to represent with equal efficiency. To achieve this, the proposed distribution will be compared with distributions studied in the literature that are based on the same foundation, utilizing a set of statistical goodness-of-fit and adequacy criteria such as: -2L, AIC, BIC, CAIC, and HQIC as well as goodness-of-fit tests like Kolmogorov–Smirnov (K-S), Anderson–Darling (A), Cramér–von Mises (W) and P-value.

Data set: which corresponds to the survival times (in years) of a group of patients given chemotherapy treatment alone for more see Gharaibeh, M. M. (2021).

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.570, 0.641, 0.644, 0.696, 0.841, 0.863, 1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

Table 4. Descriptive Statistics of Real Data Application.

Statistic	Value
Sample Size (n)	45
Mean	1.3414
Median	0.841
Mode	0.047
Variance	1.554
Standard Deviation	1.2466
Minimum	0.047
Maximum	4.033
Range	3.986

Table 4. Descriptive Statistics of Real Data Application.

Statistic	Value
Skewness	0.9721
Kurtosis	2.6638

Table 5. Adequacy criteria

Model	AIC	CAIC	BIC	HQIC
KMEBRay	121.2741136	121.8594795	126.6941011	123.2946305
BeR	123.3957638	123.9811297	128.8157513	125.4162807
KuR	122.0538534	122.6392192	127.4738408	124.0743703
EGR	123.4000829	123.9854488	128.8200704	125.4205998
WeR	122.247434	122.8327999	127.6674215	124.2679509
GoR	150.5141866	151.0995525	155.9341741	152.5347035
MOR	129.5241801	129.8098944	133.1375051	130.8711914
R	157.8332524	157.9262757	159.6399149	158.5067581

Table 6. Statistical goodness-of-fit criteria.

W	A	-2L	KS_Statistic	p_value
0.076973683	0.516139965	57.63705681	0.10139225	0.705898284
0.122518807	0.799491373	58.69788192	0.133510719	0.366182323
0.089562292	0.594676477	58.02692669	0.112796921	0.576924303
0.123364955	0.804856915	58.70004145	0.133975114	0.362059352
0.081296997	0.543627375	58.12371701	0.109418795	0.614921404
0.109842575	0.731732306	72.25709331	0.280186364	0.001287055
0.080404027	0.541649518	62.76209007	0.146021766	0.265711332
0.128784583	0.837754019	77.91662622	0.353085695	0.00001555

Form Tables (5, 6). Demonstrate that the new distribution outperforms the other distributions under comparison; it achieved the lowest values for the -2L, AIC, BIC, CAIC, HQIC, K-S, A and W. while, the new distribution have the highest p-values. This indicates the superiority of the proposed distribution over the others based on goodness-of-fit criteria, marking it as more flexible than the alternative distributions.

7. Conclusion

This study introduces a new family of statistical distributions, providing a flexible framework for generating continuous probability distributions capable of representing diverse data patterns. The derived mathematical and statistical properties demonstrate the validity of the family's probabilistic construction and allow for a multifaceted analysis of its behavior. An examination of a specific distribution derived from this family reveals that varying parameter values leads to distinct changes in the shapes of the probability density and cumulative distribution functions, reflecting the distribution's flexibility in representing varying degrees of concentration, dispersion, and asymmetry. Entropy and information-theoretic measures provide further insight into the distribution's behavior specifically regarding uncertainty and density concentration and how these are influenced by parameter values and the entropy index. In terms of application, comparisons with other distributions based on the same baseline distribution demonstrate this family's ability to yield a distribution that better fits real-world data, as evidenced by goodness-of-fit criteria. Overall, the theoretical, numerical, and applied results confirm that this new family constitutes a valuable addition to the field of continuous distributions, and that the derived distribution possesses the flexibility to serve as a suitable choice for modeling data with diverse probabilistic characteristics.

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