Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: al-kindipublisher.com/index.php/jmss

JMSS
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| RESEARCH ARTICLE

Truncated Clique Reconstruction in Non-Commuting Graphs of Finite Groups with Abelian Centralizers

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| ABSTRACT

Purpose: This study investigates how much truncated clique information is sufficient to reconstruct the non-commuting graph of a finite group with abelian centralizers. Methodology: The complete multipartite structure of these graphs is encoded by a centralizer partition profile. Clique counts are converted into power sums through Newton identities, and the multiplicities of prescribed part sizes are recovered from a Vandermonde system. The framework is then combined with arithmetic realizability conditions and exterior-square geometry for an exceptional class-three AC-3-group branch. Findings: If the possible part sizes assume d distinct prescribed values, their multiplicities are uniquely determined by the first d power sums, equivalently by a short initial segment of clique data. In the exceptional branch with $[P:Z(P)]=3^8$, $[C:Z(P)]=3^4$ and $C=P'Z(P)$, the possible non-distinguished centralizers have relative orders 3 and 9. Under the nondegeneracy condition that the associated projective kernel line is not contained in the Klein quadric, the number m_2 of relative-order-9 centralizers satisfies the sharp bound $m_2\leq 162$. An explicit AC-3-group of order 3^{12} attains equality, with non-commuting graph Γ_P ≅ K_{6480,(162)^2592,(648)^162} and ω(Γ_P)=χ(Γ_P)=2755. Significance: The results connect partial graph reconstruction, moment methods, and group-theoretic realizability, and show that truncated clique data can recover substantial algebraic-combinatorial structure.

| KEYWORDS

Non-commuting graph; group with abelian centralizers; complete multipartite graph; clique polynomial; graph reconstruction; centralizers.

2020 Mathematics Subject Classification: 05C25, 20D15, 20D60.

| ARTICLE INFORMATION

ACCEPTED: 08 August 2026 **PUBLISHED:** 24 September 2026 **DOI:** 10.32996/jmss.2026.7.6.7

1. Introduction

The interaction between finite group theory and graph theory provides a useful mechanism for encoding algebraic information in combinatorial form. One of the most extensively studied examples is the non-commuting graph of a nonabelian finite group G. Its vertex set is G \ Z(G), and two distinct vertices are adjacent precisely when they do not commute (Abdollahi, Akbari, & Maimani, 2006).

A central question asks how much information about G can be recovered from its non-commuting graph. Recent work continues to examine graph isomorphism and nilpotency questions, particularly for groups whose noncentral element centralizers are abelian (Grazian & Monetta, 2023). The present paper approaches reconstruction from a different direction: rather than assuming that the entire graph is known, it asks how much truncated clique information is sufficient to recover the graph.

This question is particularly natural for AC-groups, namely finite groups for which the centralizer of every noncentral element is abelian. In this setting the non-commuting graph is complete multipartite, and its parts are determined by the distinct noncentral centralizers. If $C_1,\dots,C_r$ are these centralizers and $z=|Z(G)|$, define $n_i=|C_i|-z$ and $P(G)=\{n_1,\dots,n_r\}$. We call P(G) the centralizer partition profile.

Our first contribution answers the reconstruction problem when the possible part sizes are known. Clique counts are elementary symmetric functions of the part sizes. Newton identities convert them into power sums, and the multiplicities are recovered from a Vandermonde system. This gives a truncated reconstruction theorem and a natural sharpness mechanism.

Not every complete multipartite profile is realizable by a finite AC-group. Group-theoretic divisibility and partition identities impose arithmetic restrictions, especially for p-groups. We therefore combine moment reconstruction with realizability conditions.

The most delicate case considered here is a class-three AC-3-group P with a distinguished centralizer C satisfying $[P:Z(P)]=3^8$, $[C:Z(P)]=3^4$, and $C=P'Z(P)$. The commutator induces a surjective alternating map $\mu:\Lambda^2(P/C)\to C/Z(P)$ with two-dimensional kernel. The geometry of this kernel inside the Klein quadric controls the possible large centralizers. Under a natural nondegeneracy condition, a fiber-packing argument gives $m_2\leq 162$, and an explicit group attaining equality is constructed.

2. Literature Review and Mathematical Background

Abdollahi et al. (2006) introduced the non-commuting graph framework in a systematic form and emphasized the problem of recovering group-theoretic information from graph structure. Their work also highlights the role of AC-groups in reconstruction questions. For AC-groups, commutativity among noncentral elements is transitive in the relevant sense, which leads to the complete multipartite structure used throughout this paper.

Rocke (1975) developed structural results for p-groups with abelian centralizers, providing classical background for the p-group branch considered here. Related transitivity phenomena for commutativity were studied by Wu (1998), placing AC/CT-type conditions in a broader structural setting.

Grazian and Monetta (2023) studied whether nilpotency is preserved under isomorphism of non-commuting graphs and obtained results specifically involving finite AC-groups. These full-graph isomorphism questions differ from the present inverse problem, which asks how much truncated clique information suffices to reconstruct the complete multipartite graph and its centralizer partition profile.

On the graph-polynomial side, clique polynomials encode clique counts in a compact algebraic form (Hoede & Li, 1994). For a complete multipartite graph, these coefficients are elementary symmetric functions of the part sizes. This observation makes Newton identities and moment reconstruction especially natural in the present setting.

3. Methodology and Reconstruction Framework

3.1 Non-commuting graphs of AC-groups

Let G be a finite nonabelian group. Its non-commuting graph Γ_G has vertex set $G \setminus Z(G)$, and x and y are adjacent if $xy\neq yx$. For $x\notin Z(G)$,

$$\deg_{\Gamma_G}(x) = |G| - |C_G(x)|.$$

A finite group G is an AC-group if $C_G(x)$ is abelian for every $x\notin Z(G)$.

Proposition 3.1. If G is a finite AC-group and $x,y\notin Z(G)$ commute, then $C_G(x)=C_G(y)$. Consequently, if $C_1,\dots,C_r$ are the distinct centralizers of noncentral elements and $z=|Z(G)|$, then

$$\Gamma_G \cong K_{\{n_1,\dots,n_r\}}, \quad n_i = |C_i| - z.$$

Proof. Since $x,y\in C_G(x)$ and $C_G(x)$ is abelian, every element of $C_G(x)$ commutes with y, so $C_G(x)\leq C_G(y)$. By symmetry equality holds. The equivalence classes of commuting noncentral elements are therefore precisely the sets $C_i\setminus Z(G)$, which are the independent parts of the non-commuting graph.

Definition 3.2. The multiset $P(G)=\{n_1,\dots,n_r\}$ is the centralizer partition profile of G. For finite AC-groups, $\Gamma_G\cong\Gamma_H$ if and only if $P(G)=P(H)$ as multisets.

3.2 Complete multipartite invariants and clique data

Let $X=K_{\{n_1,\dots,n_r\}}$ and $N=\Sigma_i n_i$. Then

$$|V(X)|=N, \quad |E(X)|=\tfrac{1}{2}(N^2-\Sigma_i n_i^2), \quad \omega(X)=\chi(X)=r, \quad \alpha(X)=\max_i n_i.$$

The clique polynomial is

$C_X(t)=\prod_{i=1}^{r}(1+n_i t)$,

so the number $c_k(X)$ of k-cliques is the elementary symmetric polynomial $e_k(n_1,\dots,n_r)$. Assume the distinct possible part sizes are $q_1<\cdots<q_d$, with multiplicities $m_1,\dots,m_d$. Define

$P_k = \Sigma_{j=1}^{d} m_j q_j^k$.

Then $P_0=\omega(X)$ and $P_1=|V(X)|$. Newton identities yield

$P_2=N^2-2|E(X)|$,

$P_3=N^3-3N|E(X)|+3c_3(X)$,

$P_4=N^4-4N^2|E(X)|+2|E(X)|^2+4Nc_3(X)-4c_4(X)$.

Thus an initial segment of clique data determines an initial segment of the moment sequence.

4. Results and Findings

4.1 Truncated clique reconstruction

Theorem 4.1 (Moment reconstruction). Let $q_1,\dots,q_d$ be distinct prescribed positive integers. Then the multiplicities $m_1,\dots,m_d$ are uniquely determined by $P_0,P_1,\dots,P_{d-1}$.

Proof. The multiplicities satisfy the Vandermonde system $V(q_1,\dots,q_d)m=P$, whose determinant is

$\det V = \prod_{i<j}(q_j-q_i) \neq 0$.

Hence the system has a unique solution.

Corollary 4.2 (Truncated clique reconstruction). For $d\geq3$, the data $\omega(X)$, $|V(X)|$, $|E(X)|$, $c_3(X),\dots,c_{d-1}(X)$ determine all multiplicities and hence X up to isomorphism. For $d=2$, $\omega(X)$ and $|V(X)|$ suffice.

The first $d-1$ moments do not generally determine d multiplicities. A kernel direction is, up to a common scalar,

$v_j = 1 / \prod_{\ell\neq j}(q_j-q_\ell)$.

For $(q_1,q_2,q_3,q_4)=(2,8,26,80)$, an integral kernel vector is $(-27,39,-13,1)$. This supplies the moment-theoretic sharpness mechanism.

4.2 Arithmetic realizability

Let $C_1,\dots,C_r$ be the distinct noncentral centralizers of a finite AC-group G, put $z=|Z(G)|$, and define $d_i=[C_i:Z(G)]$ and $q=[G:Z(G)]$. Then d_i divides q, and the partition of the noncentral elements gives

$\Sigma_{i=1}^{r}(d_i-1)=q-1$.

If $G/Z(G)$ is a p-group, write $d_i=p^{a_i}$ and $q=p^n$. Then

$\Sigma_i(p^{a_i}-1)=p^n-1, \quad \text{hence } r\equiv1 \pmod p$.

These constraints provide an arithmetic filter between abstract multipartite reconstruction and group-theoretic realizability.

4.3 Exceptional AC-3-group branch

Let P be a finite AC-3-group of class 3, let $Z=Z(P)$, and let C be the distinguished noncentral centralizer satisfying $[P:Z]=3^8$ and $[C:Z]=3^4$. We focus on the branch $C=P'Z$. Set $V=P/C$ and $A=C/Z$. Then $V\cong A\cong F_3^4$. The commutator induces a surjective alternating map $\mu:\Lambda^2V\to A$, so $K=\ker\mu$ has dimension 2.

For every non-distinguished centralizer D, the injection $D/Z\to P/C$ has image $U_D\leq V$. Since D is abelian, $\Lambda^2U_D\subseteq K$. Hence $\dim U_D\leq2$ and $[D:Z]\in\{3,9\}$. If m_1,m_2 count these centralizers, the centralizer partition gives

$80 + 2m_1 + 8m_2 = 6560, \quad \text{equivalently } m_1 + 4m_2 = 3240$.

Thus, with $z=|Z|$,

$\Gamma_P \cong K_{80z,(2z)^{3240-4m_2},(8z)^{m_2}}, \quad \omega(\Gamma_P)=3241-3m_2$.

4.4 Klein geometry and extremal rigidity

Assume $P(K)\not\subset Q$, where Q is the Klein quadric. A relative-order-9 centralizer determines a two-dimensional subspace $U\leq V$ with $[\Lambda^2U]\in P(K)\cap Q$. Since P(K) is a projective line not contained in Q, at most two admissible planes occur.

Lemma 4.3 (Fiber packing). Let Q_0 be finite, $H\triangleleft Q_0$, and $\pi:Q_0\to Q_0/H$. Fix $1\neq U\leq Q_0/H$. Suppose $L_1,\dots,L_s\leq\pi^{-1}(U)$ satisfy $L_i\cap H=1$, $\pi(L_i)=U$, and $L_i\cap L_j=1$ for $i\neq j$. Then $s\leq|H|$.

Proof. The sets $L_i\setminus\{1\}$ are pairwise disjoint and lie in $\pi^{-1}(U)\setminus H$. Since $|L_i|=|U|$, $s(|U|-1)\leq|H||U|-|H|=|H|(|U|-1)$.

Theorem 4.4 (Extremal rigidity). Under $P(K)\not\subset Q$, $m_2\leq162$. If equality holds, exactly two admissible planes U_1,U_2 occur and

$V=U_1\oplus U_2,\quad K=\Lambda^2U_1\oplus\Lambda^2U_2,\quad \mu|_{U_1\otimes U_2}:U_1\otimes U_2\cong A.$

The equality case therefore yields the two-plane structural decomposition above; the numerical multiplicities for the exceptional branch are derived immediately below and realized explicitly in Theorem 5.3.

Proof. First note that Proposition 3.1 implies that any two distinct noncentral centralizers of the AC-group P intersect exactly in $Z(P)$. Indeed, if a noncentral element x belonged to two such centralizers D_1 and D_2, then Proposition 3.1 would give $C_P(x)=D_1=D_2$. Hence, for every non-distinguished centralizer D, one has $D\cap C=Z(P)$, and for distinct non-distinguished centralizers D_i,D_j one has $D_i\cap D_j=Z(P)$. Therefore, after quotienting by $Z(P)$, the subgroups used below satisfy precisely the intersection hypotheses of Lemma 4.3. For a fixed admissible plane apply fiber packing with $Q_0=P/Z$ and $H=C/Z$. Since $|H|=81$, each plane supports at most 81 centralizers. There are at most two admissible planes, so $m_2\leq162$. Suppose equality holds. Then both admissible planes support 81 centralizers. If U_1 and U_2 met in a line, write $U_1=\langle x,y\rangle$ and $U_2=\langle x,z\rangle$. Every linear combination $\alpha(x\wedge y)+\beta(x\wedge z)=x\wedge(\alpha y+\beta z)$ would be decomposable, forcing $P(K)\subseteq Q$, a contradiction. Hence $U_1\cap U_2=0$ and $V=U_1\oplus U_2$. Moreover, Λ^2U_1 and Λ^2U_2 are distinct one-dimensional subspaces of the two-dimensional kernel K; consequently $K=\Lambda^2U_1\oplus\Lambda^2U_2$. In the decomposition $\Lambda^2V=\Lambda^2U_1\oplus(U_1\otimes U_2)\oplus\Lambda^2U_2$, the kernel is therefore disjoint from the four-dimensional cross term $U_1\otimes U_2$. Thus μ is injective on $U_1\otimes U_2$; since $\dim(U_1\otimes U_2)=4=\dim A$, this restriction is an isomorphism. In the exceptional branch of Section 4.3, the partition identity $m_1+4m_2=3240$ then gives $(m_1,m_2)=(2592,162)$, and consequently $\omega(\Gamma_P)=\chi(\Gamma_P)=3241-3m_2=2755$. These numerical consequences will be realized explicitly in Theorem 5.3.

5. Explicit Extremal Construction

Let $f(X)=X^4+X^3+X^2+1\in F_3[X]$. It has no root in F_3, and $\gcd(f,X^9-X)=1$. Hence it has neither a linear nor a quadratic factor and is irreducible. Set $F_{81}=F_3[\theta]/(f(\theta))$, with ordered basis $(1,\theta,\theta^2,\theta^3)$, so $\theta^4=2+2\theta^2+2\theta^3$.

Put $V=A=T=F_3^4\cong F_{81}$ and define $\beta(a,v)=av$ using field multiplication. Relative to $(e_1\wedge e_2,e_1\wedge e_3,e_1\wedge e_4,e_2\wedge e_3,e_2\wedge e_4,e_3\wedge e_4)$, define the alternating map μ by the matrix

M_μ = [[2,0,0,2,0,0],[0,1,0,0,0,0],[0,1,0,1,0,1],[0,1,0,0,1,0]].

Its two decomposable kernel directions correspond to $U_1=\langle e_1,e_4\rangle$ and $U_2=\langle e_2+e_4,e_1+e_3\rangle$.

Define $F(v,w)=\Sigma_{i<j}v_iw_j\mu(e_i,e_j)$, so $F_1=2v_1w_2+2v_2w_3$, $F_2=v_1w_3$, $F_3=v_1w_3+v_2w_3+v_3w_4$, and $F_4=v_1w_3+v_2w_4$. Let $G:V\times V\to T$ be the explicit polynomial map in Appendix A and define

$(v,a,t)(w,b,s)=(v+w,\ a+b+F(v,w),\ t+s+\beta(a,w)+G(v,w)).$

The exact identity $G(w,u)-G(v+w,u)+G(v,w+u)-G(v,w)=\beta(F(v,w),u)$ holds for all $v,w,u\in V$. Substitution into the multiplication law shows that this identity is precisely the associativity condition. Hence the displayed operation defines a group P on $V\times A\times T$, and $|P|=|V||A||T|=3^{12}$.

5.1 Structural and AC verification

Proposition 5.1 (Structural verification). Let P be the group defined by the multiplication law above and let $C=\{ (0,a,t): a\in A, t\in T \}$. Then $Z(P)=\{ (0,0,t):t\in T \}\cong C_3^4$, $P'Z(P)=C$, $\gamma_3(P)=Z(P)$, and $cl(P)=3$.

Proof. Put $\gamma(v,w)=G(v,w)-G(w,v)$, and use the commutator convention $[x,y]=xyx^{-1}y^{-1}$. Since $F(v,w)-F(w,v)=\mu(v,w)$, comparison of xy and yx for $x=(v,a,t)$ and $y=(w,b,s)$ gives the exact commuting criterion $\mu(v,w)=0$ and $aw-bv+\gamma(v,w)=0$ in F_{81}. The formulas in Appendix A are normalized, so $G(0,w)=G(w,0)=0$. If x is central and $v\neq0$, taking $y=(0,1,0)$ violates the second commuting equation; hence $v=0$. If $v=0$ and $a\neq0$, taking $y=(1,0,0)$ gives $aw=a\neq0$. Thus $Z(P)=\{(0,0,t):t\in T\}$. The subgroup C is abelian and $P/C\cong V$ is abelian, so $P'\leq C$. Modulo $Z(P)$, commutators have A-coordinate $\mu(v,w)$; since μ is surjective, $P'Z(P)/Z(P)=A$ and therefore $P'Z(P)=C$. For $c=(0,a,0)\in C$ and $y=(w,b,s)$, direct substitution in the multiplication law, with the stated commutator convention, gives $[c,y]=(0,0,aw)$. Field multiplication $F_{81}\times F_{81}\to F_{81}$ is surjective, so $[C,P]=T=Z(P)$. Finally, because $C=P'Z(P)$ and $Z(P)$ is central, $[C,P]=[P'Z(P),P]=[P',P]=\gamma_3(P)$. Hence $\gamma_3(P)=T$ and $\gamma_4(P)=[T,P]=1$. Therefore $cl(P)=3$.

Lemma 5.2 (Exact centralizer and AC criterion). Let $x=(v,a,t)\notin Z(P)$. If $v=0$, then $a\neq0$ and $C_P(x)=C$, which is abelian. If $v\neq0$, put $R_v=\ker(w\mapsto\mu(v,w))$. Then

$C\_P(x)=\{ (w,b,s): w\in R\_v,\ b=(aw+\gamma(v,w))/v,\ s\in T \}$. Two elements of this centralizer with V-components w_1,w_2 commute if and only if $\mu(w_1,w_2)=0$ and $\gamma(v,w_1)w_2-\gamma(v,w_2)w_1+v\gamma(w_1,w_2)=0$.

Proof. The description follows directly from Proposition 5.1: $xy=yx$ is equivalent to $\mu(v,w)=0$ and $aw-bv+\gamma(v,w)=0$. Since v is a nonzero element of F_{81}, division by v gives the displayed value of b. For $y_i=(w_i,b_i,s_i)$ in $C\_P(x)$, the commuting criterion for y_1,y_2 is $\mu(w_1,w_2)=0$ and $b_1w_2-b_2w_1+\gamma(w_1,w_2)=0$. Substituting $b_i=(aw_i+\gamma(v,w_i))/v$ and multiplying by v cancels the two terms aw_1w_2 and aw_2w_1, yielding exactly the displayed obstruction. Thus the criterion is necessary and sufficient, not merely sufficient.

For the explicit μ, 64 nonzero vectors v have dim $R\_v=1$. Then $R\_v=\langle v\rangle$, so $\mu(w_1,w_2)=0$ for $w_1,w_2\in R\_v$, and substitution $w_1=\lambda v$, $w_2=\nu v$ makes the obstruction vanish by skew-symmetry. The remaining 16 nonzero vectors are exactly $(U_1\cup U_2)\setminus\{0\}$; for them R_v equals U_1 or U_2. Hence $\mu(w_1,w_2)=0$ throughout R_v because $\Lambda^2U_i\subseteq K$. The remaining obstruction was evaluated exactly for every ordered pair in R_v^2: $16\cdot 9^2=1296$ cases, with no failures. Therefore every noncentral centralizer is abelian and P is an AC-3-group.

Theorem 5.3 (Extremal existence). The group P has order 3^{12}, class 3, and is an AC-group satisfying $[P:Z(P)]=3^8$, $[C:Z(P)]=3^4$ and $C=P'Z(P)$. Moreover $(m_1,m_2)=(2592,162)$. Consequently,

$\Gamma\_P \cong K\_\{6480,(162)^{2592},(648)^{162}\}$.

Proof. The structural assertions follow from Proposition 5.1 and Lemma 5.2. For v in an exceptional plane $U_i\setminus\{0\}$, Lemma 5.2 shows that $C\_P(v,a,t)/Z(P)$ projects isomorphically onto U_i and therefore has relative order 9. Fix one nonzero $v\in U_i$. As a ranges over $A=F_{81}$, the exact centralizer formula gives 81 distinct centralizers: if $a\neq a'$, then at any fixed nonzero $w\in U_i$ the corresponding b-values differ by $(a-a')w/v\neq 0$. Thus each admissible plane supports at least 81 relative-order-9 centralizers. Lemma 4.3 gives the matching upper bound of 81 centralizers over a fixed plane. Hence each of U_1 and U_2 supports exactly 81, and $m_2=162$. The partition identity $m_1+4m_2=3240$ then gives $m_1=2592$. Since $z=|Z(P)|=81$, the part sizes are $80z=6480$, $2z=162$, and $8z=648$, which yields the displayed graph. Hence $|V(\Gamma\_P)|=531360$, $\omega(\Gamma\_P)=\chi(\Gamma\_P)=2755$, $\alpha(\Gamma\_P)=6480$, $\delta(\Gamma\_P)=524880$, and $\Delta(\Gamma\_P)=531198$.

6. Discussion

The reconstruction theorem separates two layers of information. At the graph-theoretic layer, prescribed part sizes and a finite initial moment sequence determine the multiplicities by linear algebra. At the group-theoretic layer, divisibility and partition identities determine which formal multipartite profiles can actually arise from finite AC-groups.

The Vandermonde inversion used in Theorem 4.1 is classical linear algebra; it is not claimed as a new algebraic technique. The contribution is its formulation as a truncated-clique reconstruction mechanism for centralizer partition profiles, together with the arithmetic realizability filter and the extremal AC-group application.

This distinction also clarifies the relationship with existing non-commuting-graph literature. Full graph isomorphism questions ask whether global graph structure preserves properties such as order or nilpotency (Abdollahi et al., 2006; Grazian & Monetta, 2023). The present results instead show that, once the support of the part-size distribution is prescribed, a much smaller amount of clique information already determines the complete multipartite graph.

The exceptional AC-3-group branch illustrates that the reconstruction framework is not merely formal. Exterior-square geometry restricts the possible centralizer types, the Klein quadric controls decomposable directions, and fiber packing converts that geometry into the sharp numerical bound $m_2\leq 162$. The explicit construction demonstrates attainability rather than only feasibility.

7. Conclusion

We have studied non-commuting graphs of finite AC-groups from a truncated reconstruction viewpoint. Their complete multipartite structure reduces the reconstruction problem to recovery of the centralizer partition profile. When the possible part sizes are known and take d distinct values, the multiplicities are determined by the first d moments, and Newton identities translate this into reconstruction from a short initial segment of clique data.

The group-theoretic setting imposes additional arithmetic restrictions. In the exceptional class-three AC-3-group branch, the kernel of the induced alternating commutator map is a projective line in $P(\Lambda^2V)$, and its intersection with the Klein quadric controls the possible relative-order-9 centralizers. Under the nondegeneracy condition $P(K)\not\subseteq Q$, fiber packing yields $m_2\leq 162$, and the explicit construction shows that equality occurs.

Natural further questions include reconstruction when the possible centralizer sizes are not prescribed, classification of formal moment solutions that are group-realizable, and the complementary case $P(K)\subseteq Q$. More generally, higher-dimensional commutator kernels suggest an interaction between centralizer profiles and linear sections of Grassmannians.

Statements and Declarations

Funding: This research received no external funding.

Conflicts of Interest: The author declares no conflict of interest.

Data Availability: No external datasets were used. The finite-field certificate is fully specified by the formulas in Appendix A and the exhaustive domains stated there.

Ethics Approval and Consent to Participate: Not applicable.

Acknowledgments: None.

Appendix A. Exact Finite-Field Certificate

For $v=(v_1,v_2,v_3,v_4)$ and $w=(w_1,w_2,w_3,w_4)$, let $G(v,w)=(G_1,G_2,G_3,G_4)$ over F_3, where

$$G_1 = 2v_4w_3^2+2v_4w_2^2+2v_3w_4^2+v_3w_1w_2+v_3v_4w_3+v_2w_3w_4+v_2w_3^2+2v_2w_1^2+v_2v_4w_2+v_2v_3w_1+v_1v_3w_2+v_1v_2w_1,$$

$$G_2 = v_3w_4^2+2v_3w_2^2+v_3w_1^2+2v_2w_4^2+2v_2w_3w_4+v_2v_3w_2+v_1w_3^2+v_1w_2^2+2v_1v_3w_1,$$

$$G_3 = 2v_4w_3^2+2v_4w_2^2+2v_4w_1w_3+2v_3w_4^2+2v_3w_1w_2+v_3w_1^2+v_3v_4w_3+2v_3v_4w_1+v_2w_4^2+v_2w_3w_4+2v_2w_3^2+v_2v_4w_2+2v_2v_3w_1+2v_1w_3w_4+2v_1w_2w_3+2v_1v_4w_3+2v_1v_3w_2+2v_1v_3w_1,$$

$$G_4 = 2v_4w_3^2+2v_4w_2w_3+2v_4w_2^2+2v_4w_1w_2+v_3w_2^2+v_3w_1^2+v_3v_4w_3+2v_3v_4w_2+2v_2w_4^2+2v_2w_3w_4+v_2w_3^2+2v_2v_4w_3+v_2v_4w_2+2v_2v_4w_1+2v_2v_3w_2+2v_1w_2w_4+2v_1v_4w_2+2v_1v_3w_1.$$

The four coordinates contain 12, 9, 18, and 18 nonzero monomials, respectively, for a total of 57. Proposition A.1 (Associativity certificate). The associativity identity was evaluated exactly for every ordered triple $(v,w,u)\in V^3$. Since $|V|=81$, this is the complete set of $81^3=531441$ triples, and no failures occurred. Proposition A.2 (AC certificate). The only nonautomatic cases of Lemma 5.2 are the 16 vectors with dim $R\_v=2$. For each such v, all ordered pairs $(w_1,w_2)\in R\_v^2$ were tested; because $|R\_v|=9$, this is exactly $16\cdot 9^2=1296$ cases, again with no failures. All calculations use exact arithmetic over F_3. These computations certify the explicit extremal construction; the reconstruction theorem and the upper bound $m_2\leq 162$ are theoretical and independent of the computation.

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