Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

Dependent Competing Risks Driven by Markov-Modulated Gamma Degradation

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| ABSTRACT

Markov-modulated gamma processes and, more generally, Markov additive subordinators are established tools for degradation and reliability analysis. This paper uses such a process in a different role: as a shared latent driver of multiple cause-specific hazards. Conditional on a finite-state continuous-time Markov environment and accumulated gamma degradation, the competing causes are specified through affine degradation loadings. We derive a killed transform system for overall survival and cause-specific subdensities, establish structural invariances of the unrestricted parameterization, and distinguish what can be learned from event-only competing-risks observations from what requires repeated degradation measurements. Event-only data may recover cause-specific degradation effects when the degradation dynamics are calibrated, but generally provide insufficient information for stable joint estimation of gamma dynamics and hazard loadings. Repeated degradation sequences yield a transition-emission hidden Markov representation with matrix-valued Laplace kernels and provide sufficient conditions for generic local recoverability under explicit rank and separation assumptions. Sparse matrix-exponential and deterministic FFT implementations are verified against analytic limiting cases and an independent particle filter. In 200-replication experiments at each of three sample sizes, calibrated-loading estimates had relative bias below 1.5%, 100% numerical convergence, and decreasing root mean squared error. The framework connects competing-risks inference with degradation calibration while making its identification restrictions explicit.

| KEYWORDS

Competing risks; gamma process; Markov additive process; latent degradation; Feynman-Kac transform; right censoring

| ARTICLE INFORMATION

ACCEPTED: 08 August 2026 **PUBLISHED:** 23 September 2026 **DOI:** 10.32996/jmss.2026.7.6.6

1. Introduction

Classical competing-risks data consist of an observed time, a cause label when failure occurs, and possibly covariates and right censoring. Dependence among latent risks cannot generally be recovered without structural assumptions. One useful strategy is to introduce a shared stochastic process that represents accumulated physiological deterioration, exposure, or engineering damage. The resulting dependence is generated mechanistically: all causes respond to the same latent trajectory, but each cause may have its own loading on that trajectory.

The present work is motivated by earlier dependent competing-risks models based on latent aging processes, including the author's doctoral research on independent and dependent risks. It is nevertheless distinct from a continuous integrated-aging construction. Here the latent damage process is a pure-jump gamma subordinator whose activity changes with a finite-state Markov environment. Its paths are nondecreasing and discontinuous, representing the cumulative effect of many small shocks and occasional larger increments.

The stochastic-process foundation is not new. Paroissin and Rabehasaina studied time to threshold failure for a Markov-modulated gamma process and associated replacement policies. Shu et al. developed a broad Markov additive degradation framework with Levy subordinators, environmental switching, reliability transforms, and lifetime moments. Our contribution therefore does not lie in introducing Markov-modulated gamma degradation or its basic matrix Laplace exponent. It lies in using the degradation trajectory as a shared driver of several cause-specific hazards, deriving the corresponding right-censored competing-risks likelihood, and characterizing the resulting identification problem.

The paper makes six contributions. First, it formulates cause-specific hazards that depend jointly on covariates, the current latent environment, and accumulated jump degradation. Second, it derives a coupled first-order system for killed Laplace transforms and recovers survival, cause-specific subdensities, and cumulative incidence functions from the transform and its derivative at zero. Third, it identifies scale, baseline-state, and label invariances and gives a normalized parameterization. Fourth, it shows analytically and computationally why event-only observations cannot generally separate gamma dynamics from degradation loadings. Fifth, it develops a calibration framework based on repeated degradation increments and gives sufficient conditions for generic local recoverability in the resulting transition-emission hidden Markov model. Sixth, it supplies independent deterministic and particle likelihood implementations for noisy longitudinal degradation measurements and verifies their agreement numerically.

2. Model specification

2.1 Markov-modulated gamma degradation

Let E_t take values in {1,...,K} and follow an irreducible continuous-time Markov chain with generator $Q=(q_{rs})$. Conditional on $E_t=r$, the additive component Z_t evolves as a gamma subordinator with Levy measure

$$\nu_r(dy) = a_r\, y^{-1} \exp(-b_r\, y)\, dy, \quad a_r>0,\ b_r>0. \quad (1)$$

The corresponding Laplace exponent is

$$\psi_r(u) = a_r \log(1 + u/b_r), \quad u\geq 0. \quad (2)$$

The joint process (E_t,Z_t) is a Markov additive process. For a suitable test function f, its generator is

$$A\, f(r,z) = \Sigma_{(s\neq r)}\, q_{rs}[f(s,z)-f(r,z)] + \int_0^\infty[f(r,z+y)-f(r,z)]\nu_r(dy). \quad (3)$$

2.2 Cause-specific hazards

For causes j=1,...,m and covariate vector X, we use the affine degradation specification

$$\lambda_j(t|E_t=r,Z_t=z,X) = \lambda_{0j}(t) \exp(\beta_j'X + \delta_{jr})(1 + \rho_j\, z), \quad \rho_j\geq 0. \quad (4)$$

Define $g_{jr}(t,X)=\lambda_{0j}(t)\exp(\beta_j'X+\delta_{jr})$. The total hazard is affine in z:

$$\lambda_\bullet(t|r,z,X) = c_r(t,X) + d_r(t,X)z, \quad (5)$$

$$c_r=\Sigma_j\, g_{jr}, \quad d_r=\Sigma_j\, \rho_j\, g_{jr}. \quad (6)$$

Conditional on the complete latent path, causes are generated through their cause-specific intensities. Marginal dependence arises because every cause responds to the same unobserved environment and accumulated degradation.

3. Transform representation

Let $p_r^\dagger(t,z|x)$ denote the killed joint density of being event-free at time t with $E_t=r$ and $Z_t=z$. Define its Laplace transform by

$$\varphi_r(t,u|x) = \int_0^\infty \exp(-uz)p_r^\dagger(t,z|x)\, dz. \quad (7)$$

Writing $\varphi=(\varphi_1,...,\varphi K)$ as a row vector and $D=\mathrm{diag}(d_1,...,dK)$, the killed forward equation yields

$$\partial_t\, \varphi = \varphi Q - \varphi\, \mathrm{diag}\{c_r + \psi_r(u)\} + (\partial_u\, \varphi)D. \quad (8)$$

Proposition 1. Assume that $c_r(t,x)$ and $d_r(t,x)$ are locally bounded, the killed first moment of Z_t is finite, and differentiation under the Laplace integral is valid. Then the killed transform satisfies equation (8), with initial condition $\varphi(0,u|x)=\exp(-uz_0)\pi$. Moreover, equations (9) and (10) recover overall survival and the cause-specific subdensities.

Proof. Apply the Feynman-Kac formula to the Markov additive generator with killing rate $c_r(t,x)+d_r(t,x)z$ and test function exp(-uz). The CTMC component gives the row-vector term φQ. For state r, the gamma jump generator applied to exp(-uz) equals $-\psi_r(u)\exp(-uz)$. The constant part of the killing rate contributes $-c_r\varphi_r$. Finally, since the Laplace transform of z times the killed density is $-\partial_u\varphi_r$, the term $-d_r z$ contributes $+d_r\partial_u\varphi_r$. Combining the four terms yields (8). Summing the killed state probabilities at u=0 gives (9); weighting the killed state density by $g_{jr}(t,x)(1+\rho_j z)$ and using the same derivative identity gives (10).

Consequently, overall survival is obtained without reconstructing the full degradation density:

$$S(t|x) = \varphi(t,0|x)1K. \quad (9)$$

Moreover, because $-\partial_u\varphi_r(t,0)$ is the killed first moment of degradation in state r, the cause-specific subdensity is

$$f_j(t|x)=\Sigma_r\, g_{jr}(t,x)[\varphi_r(t,0|x)-\rho_j\, \partial_u\varphi_r(t,0|x)]. \quad (10)$$

The cumulative incidence function is $F_j(t|x)=\int_0^t f_j(s|x)ds$. The representation obeys the diagnostic identity $-S'(t|x)=\Sigma_j\, f_j(t|x)$.

4. Likelihood with right censoring

For subject i, let T_i be the observed time, Δ_i the event indicator, J_i the cause when $\Delta_i=1$, and X_i the covariate vector. Under independent censoring, the individual contribution is

$$L_i(\Theta)=S(T_i|X_i)^{(1-\Delta_i)} \prod_j f_j(T_i|X_i)^{[\Delta_i I(J_i=j)]}. \quad (11)$$

The log-likelihood follows by summation. Numerical optimization is carried out on unconstrained transformed parameters, with positivity imposed through logarithmic maps and ordered state signatures imposed through positive gap parameterizations.

5. Identification restrictions

5.1 Structural invariances

Three exact invariances must be removed. First, damage-scale transformations leave the observed model unchanged: for $c>0$, $Z^*=cZ$, $b_r^*=b_r/c$, and $\rho_j^*=\rho_j/c$ imply $\rho_j^*Z^*=\rho_jZ$. Second, multiplying the cause-specific baseline by $\exp(-\kappa_j)$ and adding κ_j to every δ_{jr} leaves g_{jr} unchanged. Third, a simultaneous permutation of latent-state labels leaves the observed law unchanged.

We use $b_1=1$ to fix the damage scale, $\delta_{j1}=0$ to define the reference state, and an ordering of mean damage rates a_r/b_r to fix state labels. At least one ρ_j must be positive, and state signatures must be distinct.

5.2 Event-only limitations

Removing exact invariances does not guarantee stable joint estimation. Event-only data may predominantly identify combinations resembling $\rho_jE(Z_t)$, while providing much weaker information about the separate gamma activity, rate, and hazard loading parameters. Numerical optimization showed likelihood ridges in which gamma variance approached zero, latent states merged, and larger degradation loadings compensated for smaller mean degradation. This motivates a distinction between calibrated event-only inference and full inference based on repeated degradation measurements.

6. Repeated degradation calibration

Suppose degradation is inspected at times $t_0<\ldots<t_L$ and increments $Y_l=Z(t_l)-Z(t_{l-1})$ are observed. Let $h_l=t_l-t_{l-1}$. Define the matrix-valued transition density $K_h(y)$ by

$$[K_h(y)]_{rs}\, dy = P(Z_{t+h}-Z_t\in dy,\ E_{t+h}=s \mid E_t=r). \quad (12)$$

Its matrix Laplace transform is the established Markov additive representation

$$\int_0^\infty \exp(-uy)K_h(y)dy = \exp\{hF(u)\}, \quad F(u)=Q-\mathrm{diag}[\psi_r(u)]. \quad (13)$$

The likelihood of one degradation sequence is

$$LD = \pi'\, K_{h1}(y_1)\, K_{h2}(y_2)\ldots K_hL(yL)1K. \quad (14)$$

Because each increment depends on both the starting and ending states, this is a transition-emission HMM rather than a standard state-emission HMM. A forward recursion with scaling constants evaluates the likelihood and posterior state probabilities.

6.1 Sufficient conditions for local recoverability

Repeated increments provide generic local recoverability when K is known, Q is irreducible, embedded transition matrices have full rank, transition-emission densities are linearly independent, normalized state signatures are distinct, and the Jacobian of observable increment distributions with respect to normalized degradation parameters has full column rank. Two or more inspection intervals improve separation and avoid relying on a single matrix logarithm. These are sufficient local conditions, not a claim that every repeated-measurement design recovers every parameter.

Once ψ_r is locally recovered, its first two derivatives at zero determine the gamma parameters: $\psi_r'(0)=a_r/b_r$ and $\psi_r''(0)=-a_r/b_r^2$, so $b_r=-\psi_r'(0)/\psi_r''(0)$ and $a_r=-[\psi_r'(0)]^2/\psi_r''(0)$.

7. Computation

The transform variable is discretized on $0=u_0<\ldots<u_M=U$. Since the characteristic velocity in the transform system is $-d_r\le 0$, a forward, second-order upwind approximation is used for $\partial_u\varphi$. Survival and cause-specific densities are extracted at $u=0$, with the same derivative order used inside the evolution equation and at the boundary. This matching is essential for discrete probability conservation.

For exponential baselines, coefficients are time invariant. The method-of-lines system can then be represented by one sparse matrix, and solutions on an equally spaced time grid are obtained by a sparse matrix exponential. Weibull and other time-varying baselines are handled by a stiff ODE solver.

8. Numerical verification and simulation

8.1 Deterministic checks

In the limiting case $\rho_1=\rho_2=0$ and $\delta_{jr}=0$, the latent process disappears and the model reduces to ordinary Weibull competing risks. The numerical survival curve agreed with the analytic solution to 1.87×10^{-11}. In a nonzero jump model, the second-order scheme achieved derivative-identity error 3.01×10^{-6} and probability-mass error 1.96×10^{-6}.

Diagnostic	Observed error
Analytic limiting survival	1.87×10^{-11}
Derivative identity, jump model	3.01×10^{-6}
Probability conservation, jump model	1.96×10^{-6}
Minimum survival and densities	Positive

8.2 Computational speed

The sparse matrix-exponential implementation reproduced the stiff ODE solver to approximately 5×10^{-11} and reduced runtime from 1.636 seconds to 0.123 seconds in the benchmark, a 13.3-fold speedup.

Quantity	BDF solver	Sparse exponential
Runtime (seconds)	1.636	0.123
Maximum survival difference	-	5.27×10^{-11}
Maximum density difference	-	4.43×10^{-11}

8.3 Calibrated-loading simulation

For each sample size, 200 independent data sets were generated with true loadings $\rho_1=0.35$ and $\rho_2=0.60$ while degradation parameters were treated as calibrated. Generation and estimation used the same refined transform grid (U=60, M=600). Optimization converged in every replicate. Relative bias remained below 1.5% in all settings, and both RMSE and empirical standard deviation declined materially as sample size increased.

n	Mean ρ_1	Bias %	RMSE	Mean ρ_2	Bias %	RMSE
300	0.3549	1.40	0.0717	0.6020	0.34	0.1194
600	0.3548	1.37	0.0536	0.6051	0.85	0.0849
1200	0.3504	0.11	0.0320	0.5977	-0.39	0.0531

Mean censoring was approximately 21.5-21.7% across scenarios, and the optimization success rate was 100% for every sample size. The empirical standard deviations closely tracked RMSE, consistent with the small observed biases.

8.4 Event-only likelihood ridge

To assess unrestricted event-only estimation, the five parameters ($\rho_1,\rho_2,a_1,a_2,b_2$) were optimized from four widely separated starting points for the same sample of n=600. The resulting parameter estimates differed radically, yet all four negative log-likelihood values lay within 0.20 of the best value. One solution sent a_2 and b_2 to tens of thousands while preserving their ratio; another increased ρ_2 above 2.7 while reducing gamma activity. These compensating solutions empirically demonstrate the weak practical identification predicted by the structural analysis.

Start	ρ_1	ρ_2	a_1	a_2	b_2	ΔNLL
1	0.193	0.512	1.292	40466.5	31309.4	0.000
2	0.214	0.572	1.001	1.473	1.000	0.059
3	1.008	2.713	0.177	0.482	1.168	0.198

Start	ρ_1	ρ_2	a_1	a_2	b_2	ΔNLL
4	0.198	0.534	0.871	83.197	43.826	0.063

For comparison, the negative log-likelihood at the data-generating parameters was only 2.64 above the best sample-specific value. The purpose of this experiment is not to recover the generating point in one finite sample, but to show that very different latent decompositions yield nearly indistinguishable observed-data likelihoods.

8.5 Noisy longitudinal likelihood check

An independent small-step bootstrap particle filter was implemented for the joint likelihood of noisy degradation visits and the terminal competing-risk outcome. It provides an independent validation device for the deterministic transition-emission recursion. In the limiting event-only model with zero degradation loadings and no state effects, the particle likelihood reproduced the analytic exponential competing-risk log-likelihood with absolute error 5.64×10^{-14}.

A second record contained nine noisy longitudinal measurements before a cause-1 event. Across eight independent particle-filter runs, increasing the number of particles from 8,000 to 32,000 reduced the empirical standard deviation of the estimated log likelihood from 0.276 to 0.178. The corresponding mean estimates were -9.905 and -10.132; their difference is compatible with the observed Monte Carlo variability and remaining time-discretization error.

Check	Result
Analytic null log likelihood	-2.9157286443
Particle null log likelihood	-2.9157286443
Absolute null error	5.64×10^{-14}
SD at 8,000 particles	0.276
SD at 32,000 particles	0.178

8.6 Deterministic transition-emission recursion

A deterministic forward filter was implemented on a damage grid. Each small time step applies the CTMC transition, a state-specific gamma convolution evaluated by FFT, exponential killing by the total competing-risk hazard, and—at visit times—the Gaussian measurement update. First-order Lie operator splitting is used; the reported grid-refinement study directly quantifies its numerical error.

With a sufficiently wide damage domain, the deterministic filter reproduced the analytic null log likelihood with absolute error 2.51×10^{-12}. For the nine-visit record, progressive refinement (dt,dz)=(0.05,0.02), (0.025,0.01), and (0.0125,0.005) produced log likelihoods -10.1015, -10.0983, and -10.0966. The finest deterministic value differed from the 32,000-particle mean (-10.1320) by only 0.0355, compared with a particle standard deviation of 0.1782.

Method or grid	Log likelihood	Runtime (s)
Deterministic: 0.05, 0.02	-10.1015	—
Deterministic: 0.025, 0.01	-10.0983	0.086
Deterministic: 0.0125, 0.005	-10.0966	—
Particle filter: 32,000	-10.1320 mean	1.426

At the medium deterministic resolution, one likelihood evaluation was approximately 16.5 times faster than one 32,000-particle evaluation. These comparisons support both the transition-emission construction and its use inside likelihood optimization, while the particle filter remains an independent validation benchmark.

9. Potential application and data requirements

A natural potential application is a kidney-transplant cohort with repeated estimated glomerular filtration rate (eGFR), time to graft failure, death with a functioning graft as the competing cause, and administrative censoring. This design supplies the three elements required by the model: longitudinal information about accumulated physiological damage, two mutually exclusive first events, and a sufficiently large number of independent recipients. It is scientifically better aligned with the likelihood than turbofan benchmarks in which failure modes may be simultaneous or the terminal cause is not recorded at unit level.

Let G_{i0} be a stable post-transplant reference eGFR and define latent accumulated damage $D_i(t)=G_{i0}-G_i$^*(t), where G_i^*(t) is the unobserved noise-free renal-function trajectory. Because clinical eGFR can improve temporarily and is measured with error, the observed marker must not be treated as an exact gamma subordinator. The application will therefore use the measurement layer

$$Y_i(t_l)=G_{i0}-D_i(t_l)+\text{epsilon}_{il}, \quad \text{epsilon}_{il} \sim N(0,\text{sigma}_e^2), \quad (15)$$

while D_i follows the Markov-modulated gamma process. Integration over latent damage increments can be performed by a scaled forward recursion combining the matrix transition-emission kernel with the Gaussian measurement density. The survival contribution remains the killed-transform likelihood developed above.

An eligible cohort should contain unit identifiers, transplant origin time, at least two post-baseline eGFR measurements for a substantial fraction of recipients, an adjudicated first-event code distinguishing graft failure from death with a functioning graft, and censoring dates. A future analysis would end follow-up at the first competing event, as required by the estimand, and would handle missing marker visits through the observation process rather than convert them into zero degradation increments.

Required field	**Role in analysis**
Recipient ID and transplant date	Independent unit and time origin
Repeated eGFR with dates	Noisy degradation observations
Graft-failure date	Cause 1 event time
Death date and graft status	Cause 2 event time
Last follow-up date	Right censoring
Age, sex, donor type	Prespecified baseline covariates

The United Network for Organ Sharing application reported by Dong et al. comprised 5,654 recipients and longitudinal GFR trajectories, with 1,590 graft failures and 1,028 deaths with a functioning graft. It establishes feasibility of the data structure, but its analytic file is not an unrestricted public download. Consequently, no empirical estimates will be reported until an eligible cohort is obtained under its applicable data-use terms. This prevents substituting a convenient but structurally incompatible public benchmark for the target likelihood.

10. Discussion

The proposed model connects two established areas: Markov-modulated degradation and dependent competing risks. Its contribution is not a new subordinator but a new inferential role for the subordinator. Accumulated jump damage induces dependence across cause-specific hazards, and the transform system converts this mechanism into computable survival and cause-specific likelihood contributions.

The identification analysis is central rather than ancillary. Complex latent-process survival models can fit observed event distributions while assigning very different decompositions to latent activity, jump scale, and hazard loading. By exposing exact invariances and practical likelihood ridges, the framework specifies which conclusions require degradation calibration and which can be obtained from event data alone.

The numerical evidence supports the chosen computation without requiring a higher-order splitting method: grid refinement was stable, the deterministic likelihood agreed with the independent particle benchmark, and the analytic limiting cases were recovered to high precision. Bayesian propagation of calibration uncertainty, covariate-dependent environmental transitions, and dynamic prediction conditional on inspection history remain possible extensions.

11. Conclusion

A Markov-modulated gamma degradation process can serve as a shared latent driver of dependent competing risks. The affine cause-specific specification yields a tractable killed-transform system and efficient likelihood computation. Event-only data support estimation of cause-specific degradation effects when the latent degradation dynamics are calibrated, but do not generally support stable unrestricted joint estimation. Repeated degradation sequences provide the additional information needed for generic local recovery under explicit rank and separation conditions. The resulting framework combines mechanistic dependence, computable inference, and transparent identification limits.

Statements and Declarations

Funding: This research received no external funding. The article processing charge, if applicable, will be funded by the author.

Conflicts of Interest: The author declares no conflict of interest.

Data Availability: The numerical results reported in this article are based on simulated data generated under the model specifications described in the manuscript. No external empirical dataset was analyzed.

Code Availability: The numerical procedures are described in sufficient detail for independent implementation. Computational code is available from the author upon reasonable request.

Ethics Approval: Not applicable. This methodological study did not involve human participants, animals, or identifiable personal data.

Author Contributions: Yousef Al-Zalzalah conceived the study, developed the methodology, conducted the analysis, and wrote and approved the manuscript.

Acknowledgments: The author thanks the editorial office for its pre-submission guidance.

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