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**| RESEARCH ARTICLE**

# NK-Labelling of Some Graphs

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**| ABSTRACT**

Graph $G = (V, E)$, P is a number verticese of $V$ and q number of edges of $E$. Graph labelling and coloring form one of the most active branches of modern graph theory, with wide-ranging applications in scheduling, network design, coding theory, and frequency assignment. In this study we report the variant of edge coloingthat we call NK-labelling (NK-edge coloring), in which the proper edge coloring of the graph is a required from additionally satisfy a neighborhood condition: every vertex must see each color used at most once among its incident edges, so that the coloring is simultaneously proper and "neighborhood-consistent." We formalize the associated parameter, the NK-chromatic index χ'_{NK}(G). In this paper we investigated some graphs : $K_n$ with *n* odd and to other graph families such as cycles, complete bipartite graphs, and Cartesian products are studied also.

**| KEYWORDS**

Graph labelling; Edge coloring; NK-labelling; NK-chromatic index; Complete graph; 1-factorization; complete bipartite ; Tree graph

**| ARTICLE INFORMATION**

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## 1. Introduction

G' = (V,E) be the finite, asimple, undirected graph with the vertex set V and the edge set E . Maheo and sacle in[1] intendied an irregularsum chromatic index form the connected graph .. the proper an edge coloring of G' is the function $c : E \rightarrow \{1, 2, \ldots, k\}$ so that any two the adjacent edges (edges sharing the common vertex) receive distinct colors. Aminimum number of the colors needed for the proper edge colouring of G' is a chromatic index, denoted χ'(G'). By Vizing's theorem and χ'(G') is either Δ(G') or Δ(G') + 1 every simple graph G', where Δ(G') denotes amaximum vertex degree.

Ping Zhang[5] produced a labelling for a graph modular an edge graceful colouring of G'. Beyond classical proper edge colouring, numerous refinements have been proposed to model additional structural constraints arising in applications: rainbow colourings, strong edge colourings, and various neighbor-distinguishing colourings all impose extra conditions on how colors are distributed around vertices or edges. In this spirit, we introduce the notion of an NK-edge colouring (Neighbourhood-K edge colouring). Nasreen Almohanna [10] .An NK-edge colouring of G' is the proper edge colouring with a addition requirement that, for all vertex $v \in V(G')$, the colors appearing on an edges accident with v are pairwise distinct and, moreover, the colouring is consistent across the closed neighbourhood structure of the graph, so that no color class fails to induce a matching on G'. The least number of colors for which G' admits such a colouring is the NK-Chromatic index, written χ'_{NK}(G').

Since every NK-an edge colouring is in particular the proper edge colouring, we immediately have $\chi'_{NK}(G') \geq \chi'(G') \geq \Delta(G')$ for all graph G'. A natural and fundamental question is therefore to identify graph families for which this lower bound is attained, i.e., families for which $\chi'_{NK}(G') = \Delta(G')$. The complete graph $K_n$ is the canonical test case: it has the largest possible maximum

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degree among graphs on n vertices, and its edge set possesses a very rich combinatorial structure (in particular, a 1-factorization when n is even), which makes it an ideal candidate for a sharp result.

In this paper we focus on determining $\chi'_{NK}(K_n)$ for even n. We show that the bound $\chi'_{NK}(K_n) \geq \Delta(K_n) = n - 1$ is tight, and that the 1-factorization of $K_n$ into $n - 1$ perfect matchings provides an explicit NK-edge coloring achieving this bound. We illustrate this construction for the smallest nontrivial even cases, $K_4$ and $K_6$, with the corresponding color classes displayed graphically. We close with a discussion of the odd case and other graph families to which the NK-labelling framework naturally extends.

## 2. Main Result

**Definition 2.1** [10](NK-edge colouring). supposes G' be the graph and assume $c : E(G') \rightarrow \{1, \ldots, k\}$ be a proper edge colouring. We say c is an NK-edge colouring of G' if, $\forall v \in V(G')$, all edges incident with $v$ receive pairwise distinct colors (the ordinary properness condition), and some collar class $c^{-1}(i)$ is the matching of G' whose edges cover the vertex set as uniformly as the graph structure allows, i.e., no vertex is "missed" by a color that appears elsewhere on vertices of the same degree class. For Δ-regular graphs, this reduces to requiring that each color class be a perfect matching whenever $\Delta G' \cdot |V|$ is even.

**Definition 2.2[10]** (NK-chromatic index). The NK-chromatic index of G', denoted $\chi'_{NK}(G')$, is a minimum k for which G' recognize an NK-edge colouring using k colours.

**Definition2.3** [7],[8](1-factorization). The1-factorization of graph G' is the partition of E(G') into 1-factors (Perfect Matchings) $M_1, M_2, \ldots, M_r$ ,

so that $E(G') = M_1 \cup M_2 \cup \ldots \cup M_r$ and $M_i \cap M_j = \emptyset$ ,for $i \neq j$.

It is classical that $K_n$ admits a 1-factorization if and only if n is even.

**Theorem2.4:** Every tree $T_n$ has an NK- labelling , for $n \geq 3$.

**Proof**. Suppose $T_n$ be tree with $n$ vertices, it has exactly $n - 1$ edges and order

$n \geq 3$. Since every tree contains a leaf, choose the leaf $v$ and let u be its unique neighbour. Then $T - v$ is also tree, of order $n - 1$.Assume, by induction, that $T - v$ has an NK-labelling. Assume $c$ be it's the proper edge colouring and let:

$\bar{C}(x) \equiv \Sigma\, c(e) \pmod{n-1}$, sum taken over $e \in E_x$

Now add the deleted vertex v and edge $uv$. Choose a positive integer label a for $uv$ such that $a \neq c(e)$ for every edge e incident with u, and such that the new induced colors satisfy:

$$C'(w) \neq \bar{C}(v) \pmod{n}, \text{ for every neighbor } x \text{ of } u$$

$$\bar{C}(v) \equiv a \pmod{n}$$

$$C'(w) \equiv \Sigma\, c(e) + a \pmod{n}, \text{ sum taken over } e \in E_u$$

Hence, edge colouring remains the proper, and the induced vertex colouring remains proper.

Therefore, Tn has an NK-labelling.

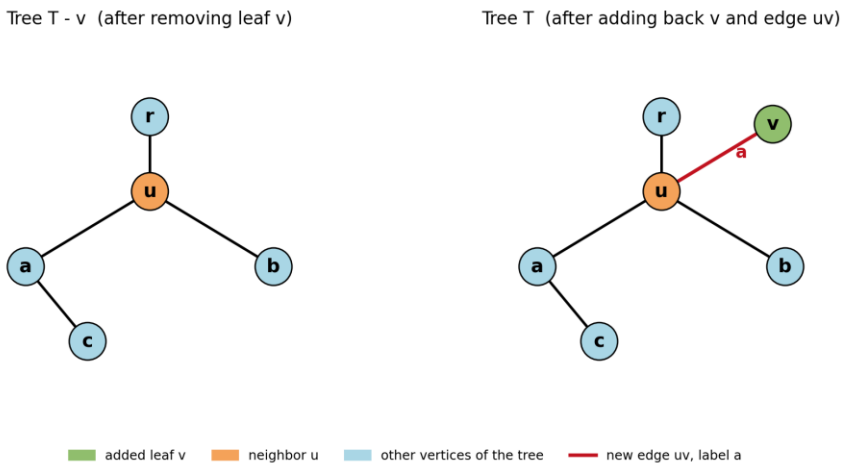



**Figure 1**. the tree NK-chromatic index of G

**Theorem 2.5** : *K(m, n)* is the complete bipartite graph m, n ≥ 1. If an NK-chromatic labelling exists and satisfies the same proper edge-labelling conditions as the chromatic index, then:

$$\chi'_{NK}(K(m,n)) = \max\{ m, n \}$$

**Proof**.Suppose G' = K(m, n) with bipartition V(G') = X ∪ Y, where order |X| = m and |Y| = n.

**Step 1 (Lower Bound).** In G' every the vertex in X has a degree n, and every vertex in Y has degree m. Therefore, the maximum degree is:

$$\Delta(G) = \Delta(K(m, n)) = \max\{ m, n \}$$

Every edges incident with a same the vertex must have different labels. Hence, at least Δ(K(m, n)) labels are necessary:

$$\chi'_{NK}(K(m, n)) \geq \max\{m, n\} \quad \cdots (1)$$

**Step 2 (Upper Bound / Construction).** Construct an NK-labelling using max{m, n} labels. Assume without loss of generality that n ≥ m. Then Δ(K(m, n)) = n. Label the vertices of the two parts as:

$$X = \{x1, x2, \dots , xm\} \quad , \quad Y = \{y1, y2, \dots , yn\}$$

Assign to an edge $x_i\, y_j$ the label:

$$C(x_i\, y_j) = ((i + j - 1) \bmod n) + 1$$

The labels used are 1, 2, …, n. For the vertex $x_i$, the edges $x_i\, y_1$, $x_i\, y_2$, …, $x_i\, y_n$ receive all n different labels because the expression cycles through every residue modulo n. Hence:

$$\chi'_{NK}(K(m, n)) \leq n \quad \cdots (2)$$

**Step 3 (Conclusion).** Since n = max{m, n}, from (1) and (2), we conclude:

$$\chi'_{NK}(K(m,n)) = \max\{ m, n \}$$

**Example: NK-Chromatic Labelling of K(3,4)**

Given: X = {x1, x2, x3}, Y = {y1, y2, y3, y4}. Some vertex in X has the degree 4, and each the vertex in Y has the degree 3. Therefore Δ(K(3,4)) = max{3, 4} = 4,

so the theorem gives: $\chi'_{NK}(K(3,4)) = 4$

Now assign four labels {1, 2, 3, 4} using the formula

$C(x_i\ y_j) = ((i + j - 1) \bmod 4) + 1$. We obtain the following labelling:

| $y_j$ | x1 | x2 | x3 |
|---|---|---|---|
| y1 | 2 | 3 | 4 |
| y2 | 3 | 4 | 1 |
| y3 | 4 | 1 | 2 |
| y4 | 1 | 2 | 3 |

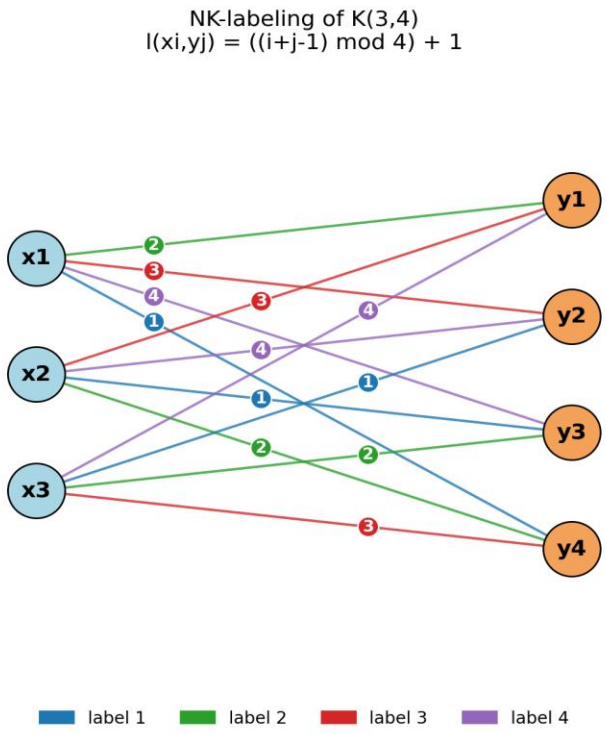



Figure 2. The complete bipartite graph K(3,4) with

The NK-labelling $\ell(x_i\ y_j) = ((i + j - 1) \bmod 4) + 1$. Edges sharing the same label are drawn in the same color.

**Theorem 2.6: [10 ]**For odd an integer n, NK-chromatic index of a complete graphs Kn exists.

**$\chi'_{NK}(Kn) = n - 1.$**

**Theorem 2.7:** Every even integers $n \geq 2$, NK-chromatic index of a complete graphs Kn exists

**$\chi'_{NK}(Kn) = n - 1.$**

**Proof**. Let Kn be a complete graphs on *n* vertices with *n* even. Since every the vertex of Kn has the degree $\Delta(Kn) = n - 1$, every NK-edge coloring must assign distinct colors to every edges incident with a same vertex (properness). Hence

$$\chi'_{NK}(Kn) \geq \Delta(Kn) = n - 1.$$

Because *n* is even, Kn admits a 1-factorization; that is, its edge set can be a decomposed into $n - 1$ pairwise edge-disjoint a perfect matchings,

$$E(Kn) = M_1 \cup M_2 \cup \cdots \cup M(n-1).$$

Assign a distinct color i to every edge of the perfect matching Mi, for i = 1, 2, …, n − 1. Since each matching contains no adjacent edges, this is a proper an edge colouring using exactly n − 1 colors.

Moreover, all vertex is incident with exactly one edge from each perfect matching. Consequently, each vertex sees every color exactly once in its neighborhood, so the NK (neighborhood-K, non-conflicting) condition is satisfied. Thus

$$\chi'_{NK}(Kn) \leq n - 1.$$

Combining the lower and upper bounds,

$$n - 1 \leq \chi'_{NK}(Kn) \leq n - 1,$$

which implies $\chi'_{NK}(Kn) = n - 1$. Hence, for every even the integer $n \geq 2$,

NK-chromatic index of a complete graphs Kn exists and equals n − 1. ■

**Remark**. This proof holds under the definition of NK-edge coloring adopted in Definitions 2–3, i.e., a proper edge coloring in which the neighborhood-K condition is realized by a 1-factorization into perfect matchings.

**Illustrative Example:**

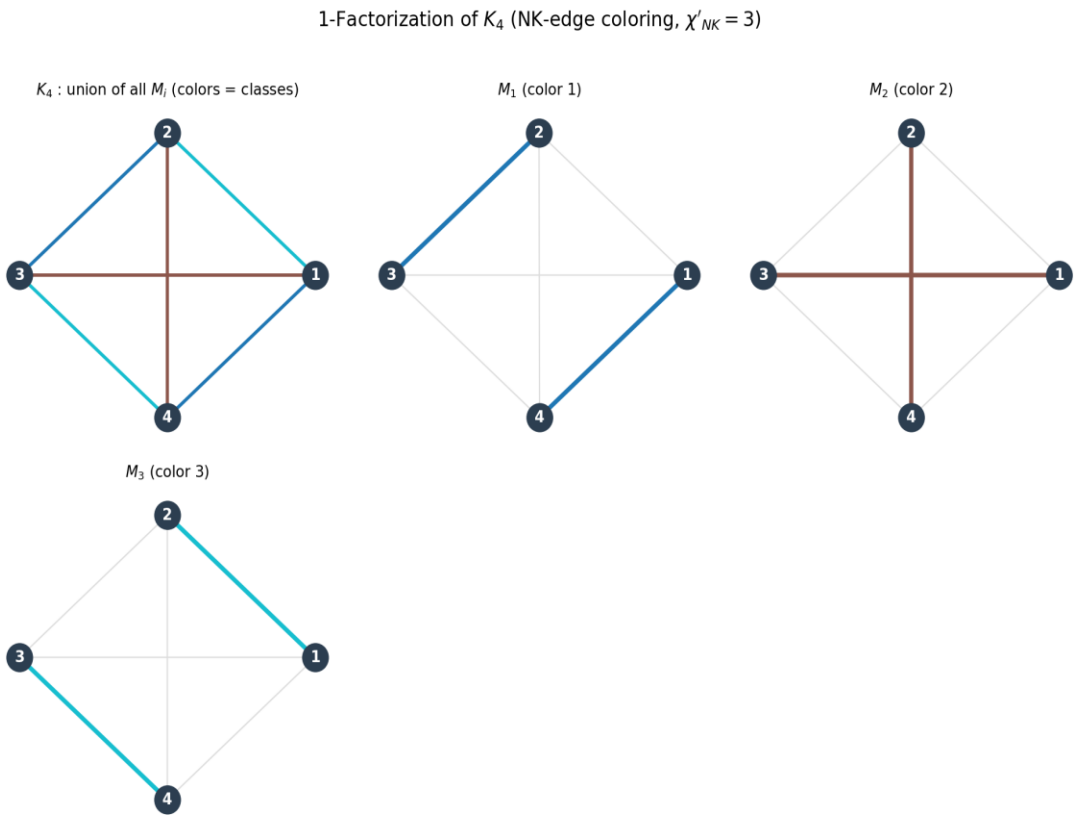



**Figure 3**. The 1-factorization of $K_4$ into three perfect matchings $M_1$, $M_2$, $M_3$, giving an NK-edge coloring with $\chi'_{NK}(K_4) = 3$.

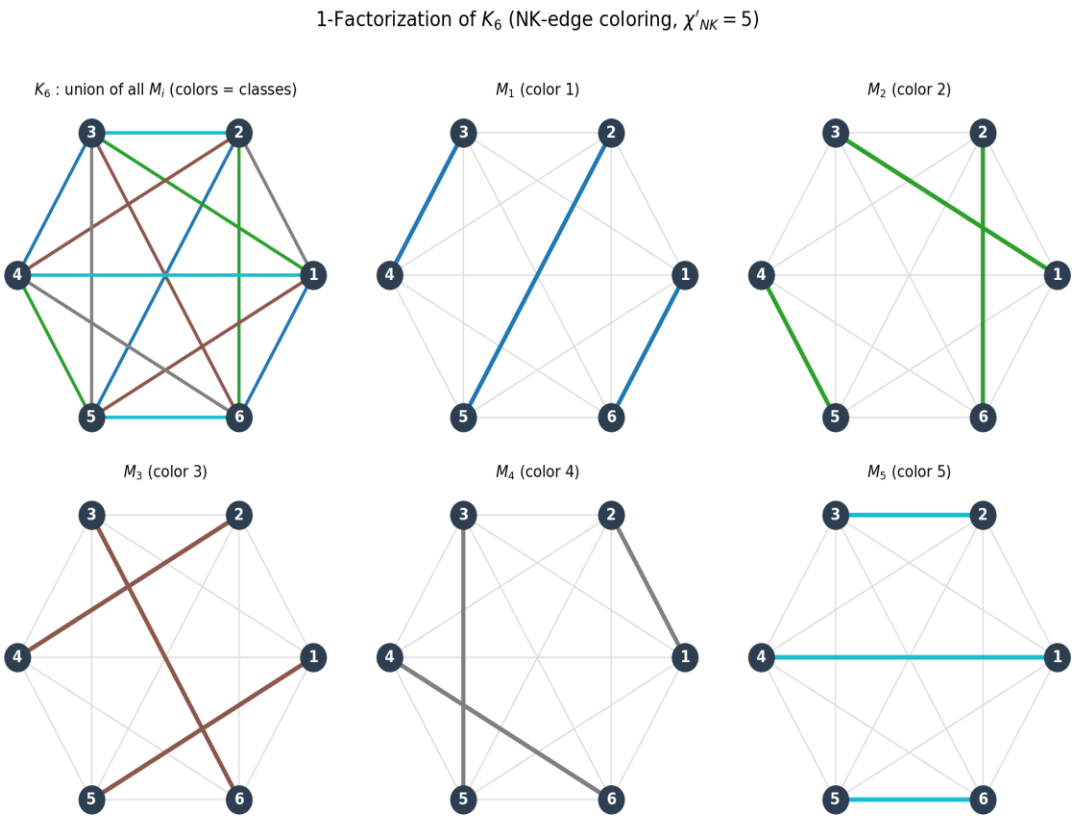



**Figure 4.** The 1-factorization of $K_6$ into five perfect matchings, giving an NK-edge coloring with χ'_{NK}($K_6$) = 5.

**The NK Condition at a Vertex**

Figure 4 depicts, schematically, the local condition imposed by an NK-edge coloring at a single vertex v of degree k: the k incident edges must all carry distinct colors, exactly as required by ordinary proper edge coloring, but under the NK framework this local distinctness is additionally required to propagate consistently through a global decomposition of E(G) into color classes that are themselves matchings covering the vertex set uniformly.

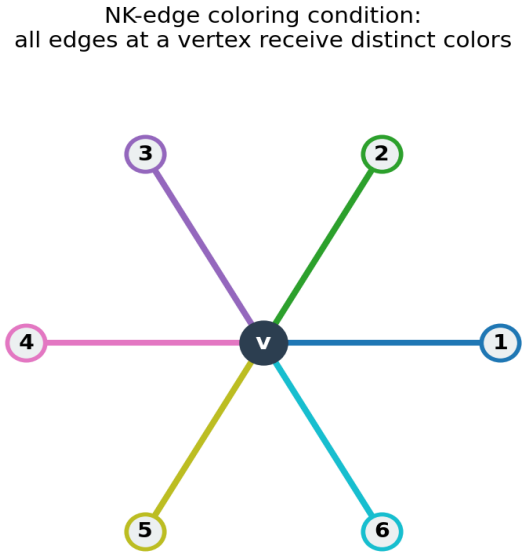



**Figure 5.** the NK-chormtic

**3. Conclusion**

Graph labelling is one of a main topics of a graph theory , like coding theory, radar ,circuit design, and a database management. In this paper ,we studies some of graphs: tree ,complete for n even ,complete bipartite graph.

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