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| RESEARCH ARTICLE

A New Goodness-of-Fit Test for the Two-Parameter Gamma Distribution

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| ABSTRACT
This paper proposes a new goodness-of-fit for the two-parameter distribution. It is based on a function of squared distances between empirical and theoretical quantiles of a set of observations being hypothesized to have come from the gamma distribution. The critical values of the proposed statistic are evaluated through extensive simulations of the unit-scaled gamma distributions and computations. The empirical powers of the statistic are obtained and compared with some well-known tests for the gamma distribution, and the results show that the proposed statistic can be recommended as a test for the gamma distribution.

| KEYWORDS
Two-parameter gamma distribution, quantile function, goodness-of-fit test, empirical critical value, empirical power of a test, distance function.

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1. Introduction

The gamma distribution is a two-parameter family of continuous probability distributions with two different parameterizations. Its first parameterization with α and β as the shape and scale parameters respectively gives rise to its probability function as $f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\}$ while the second parameterization of α and θ as the shape and rate parameters respectively gives rise to $f_X(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp\{-x\theta\}; x, \alpha, \beta, \theta > 0$, where *X* is the gamma random variable and $\theta = 1/\beta$. As a family of distributions, it collapses into different known probability distributions with different values of the parameters in the two parameterizations. For instance, if α equals 1, the gamma distribution becomes equal to the exponential distribution. Other distributions that belong to the family include the Erlang distribution and the chi-square distribution.

Because of its role as parent distribution to many continuous probability distributions, it has a very wide application in probability theory as well as in statistics and engineering. For instance, it is used in life testing, waiting time modelling and generally in reliability studies. Also, it is used in Bayesian statistics as a conjugate prior distribution. Again, it is used extensively in communication engineering due to its property of distribution with maximum probability entropy. Due to this wide application of the gamma distribution in sciences, it is important to determine how good a dataset that is assumed to be gamma-distributed is in actuality to the gamma distribution before the intended use. This is because using a dataset that is not from a gamma distribution with the wrong assumption of the dataset being gamma-distributed may give rise to a wrong result with far dare consequences.

Mitigating this type of problem so far raised has prompted a number of researchers to delve into the development of goodness-of-fit statistics for testing whether or not a dataset comes from the gamma distribution by employing different theoretical properties of the gamma distribution. For instance, Kolmogorov (1933) and Anderson and Darling (1954) have used different distance functions between the theoretical and empirical distribution functions of the gamma distribution to develop some time-honoured tests for the gamma distribution. Some other tests for the gamma distribution in the literature include Locke (1976); Romantsova (1996); Castillo and Puig (1997); Kallioras, Koutrouvelis and Canavos (2006); Wilding and Mudholkar (2008) and Henze, Meintanis and Ebner (2012).

Generally, in goodness-of-fit tests, the various distance functions between theoretical and empirical unique functions have dominated the research area, obviously due to its tractability as well as high power performances. These functions include the distribution function, the characteristic function, the moment generating function, the Laplace function and the quantile function. Unfortunately, at least, to the best of the knowledge of the researcher, no test for gamma distribution has been proposed in the literature with the use of the quantile function despite its uniqueness property. This is the main thrust for this work. The rest of the paper is organized as follows: The statistic is proposed in section two, while the empirical critical values, as well as empirical comparison, is given in section three. The work is concluded in section four.

2. The Test Statistic

The population (theoretical) quantile function, $Q_X(p);\ p \in (0,1)$, of a random variable *X* with distribution function, $F_X(x)$, and density function, $f_X(x)$, is defined by:

$$Q_X(p) = F_X^{-1}(x);\ p \in (0,1) \tag{1}$$

From (1), it is clear that it can be called the inverse distribution function. In fact, Jones (1992) uses it to present the quantile function as:

$$Q_X(p) = \inf\{x : F_X(x) \geq p\};\ p \in (0,1) \tag{2}$$

And states that it can be for both discrete and continuous random variables. For a continuous random variable, *X*, (2) becomes:

$$Q_X(p) = \inf\{x : F_X(x) = p\};\ p \in (0,1) \tag{3}$$

Now, suppose *X* has a two-parameter gamma distribution with parameters α and β, and a probability density function,

$f_X(x,\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\};\ x, \alpha, \beta > 0$. The distribution function is obtained by:

$$F_X(x) = \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \tfrac{x}{\beta}\right) \tag{4}$$

where $\gamma\left(\alpha, \frac{x}{\beta}\right)$ is the lower incomplete gamma function. This, therefore, gives rise to the non-closed form expression for the distribution function of the gamma distribution. By the relationship between quantile function and distribution function of a random variable, as seen in (1), the quantile function of the two-parameter gamma random variable *X* is given by:

$$Q_X(p) = \Gamma(\alpha)\gamma^{-1}\left(\alpha, \tfrac{x}{\beta}\right);\ p \in (0,1) \tag{5}$$

where $\gamma^{-1}\left(\alpha, \frac{x}{\beta}\right)$ is the inverse lower incomplete gamma function.

From (5), it is obvious that the exact quantile measure of the gamma random variable for a given probability value, *p*, is almost intractable. Abramowitz and Stegun (1972) have obtained an approximation to the incomplete gamma function, and Shea (1988) has obtained a computer algorithm for the approximation. Based on this approximation, a number of authors have approximated the quantile function of the gamma distribution. This has also been implemented in several computer packages for statistical analysis, such as R. An example of such approximations is Okagbue, Adamu and Anake (2020). This, therefore, means that the quantile function (or quantiles) of a gamma random variable can serve as a unique property of it, notwithstanding its approximate measure.

The quantile function of a random variable can be estimated from a random sample drawn from the population of the random variable. Xu and Miao (2011) state that the *p*th quantile of a distribution can be estimated by either the sample *p*th quantile of the distribution or the appropriate *k*th order statistic of a sample drawn from the distribution. This amounts to estimating a population parameter by either of two statistics, which are of different concepts. The sample *p*th quantile of a distribution, denoted by $Q_n(p)$ is obtained as the inverse of the sample distribution function, also known as the empirical distribution function, which is denoted by $F_n(x)$. For $p \in (0,1)$,

$$Q_n(p) = F_n^{-1}(p) = \inf\{x : F_n(x) \geq p\} \tag{6}$$

where $F_n(p) = n^{-1}\sum_{i=1}^{n} I(X \le x)$ is the average number of observations in the random sample that are less than or equal to x. Let the number of observations in the sample that are less than or equal to x be *i*. Then $F_n(x) = i/n$. Hence, $p \in (0,1)$ can be approximated by i/n. With this understanding, a random sample of size *n* drawn from a non-negative continuous random variable, when ordered as $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$, where $X_{i:n}$ is the *i*th smallest observation in the sample, gives for $p = i/n$ $F_n^{-1}(\frac{i}{n}) = X_{i:n}$. More concretely, Serfling (1980) has shown that provided $i/n \to p$, $\left|X_{i:n} - F_n^{-1}(p)\right| \xrightarrow{a.s.} 0$. This, of course, settles the use of either the sample *p*th quantile or the *k*th order statistics as appropriate estimators of the theoretical quantile function. What is left therefore is to obtain a distance function $D\left(X_{i:n}, Q_X(p)\right)$ for which $p \in (0,1) = p \in \left(\frac{i-1}{n}, \frac{i}{n}\right); i = 1, 2, \ldots, n$, which measures the distance apart between the *p*th sample quantile and the *p*th theoretical quantile with the understanding that this distance will tend to zero if the sample is obtained from the gamma distribution.

To obtain this, each observation has to be rescaled to one so as to obtain a statistic that is not dependent on the scale parameter β. Let a random sample whose distribution is to be determined be $x_1, x_2, \ldots, x_n$. Obtain an estimator of β as $\hat{\beta}$. Then, the rescaled observations are obtained as $y_1, y_2, \ldots, y_n$ where $y_i = \frac{x_i}{\hat{\beta}}; i = 1, 2, \ldots, n$ and the distance function is between each $Y_{i:n}$ and the theoretical quantile of the gamma distribution with $\beta = 1$ where $Y_{i:n}$ is the *i*th smallest observation in the rescaled sample. Note that the consistent bias-corrected estimator for β is given by:

$$\hat{\beta} = \frac{1}{n(n-1)}\left(n\sum_{i=1}^{n} x_i \ln(x_i) - \sum_{i=1}^{n}\ln(x_i)\sum_{i=1}^{n} x_i\right) \tag{7}$$

Therefore, with the average of the interval, $p \in \left(\frac{i-1}{n}, \frac{i}{n}\right); i = 1, 2, \ldots, n$, for each *i* as $p_i = \frac{i-0.5}{n}; i = 1, 2, \ldots, n$ and following from Madukaife (2019), an appropriate test for the gamma distribution is obtained by:

$$D_n = \sum_{i=1}^{n}\left(Y_{i:n} - Q_Y\left(\tfrac{i-0.5}{n}\right)\right)^2 \tag{8}$$

where $Y_{i:n}$ is the *i*th order statistic of the rescaled dataset and $Q_Y\left(\frac{i-0.5}{n}\right)$ is the *p*th theoretical quantile of the unit scaled gamma distribution with $p = \frac{i-0.5}{n}$. The proposed statistic rejects the null hypothesis of the gamma distribution for large values of D_n.

3. Simulation Studies

In this section, the empirical critical values, as well as the empirical power comparison, are considered.

3.1 Critical values of the statistic

There is no effort in this paper to derive the exact or asymptotic distribution of the D_n statistic. As a result, it will not be possible at this point to obtain the theoretical, critical values of the test. In this work, therefore, the critical values are obtained through extensive simulations as empirical critical values. The only disadvantage of this is that no matter how powerful this statistic may be, its usefulness can only be with the use of statistical software. Fortunately, there is easily available software for the use of anyone who is interested. The empirical critical values thus obtained in this section, therefore, is only for the purpose of demonstration.

The empirical critical values of the proposed statistic are evaluated in this paper at two levels of significance (alpha = 0.01, 0.05), four values of the shape parameter (α = 1, 2, 3, 4) and nine sample sizes (*n* = 5, 10, 15, 20, 25, 30, 35, 40 and 50). In each sample size and shape parameter situation, 100,000 samples are generated from the unit-scaled gamma distribution with the appropriate shape parameter and the value of the statistic is evaluated from each generated sample, giving rise to a total of 100,000 evaluated statistics. The alpha-level critical value for such a situation is therefore obtained as the 100(1 - alpha) percentile of the evaluated statistics. The empirical critical values are presented in Table 1.

Table 1. Empirical critical values of the D_n statistic, alpha = 0.01, 0.05.

	Alpha = 0.05				Alpha = 0.01			
n	α = 1	α = 2	α = 3	α = 4	α = 1	α = 2	α = 3	α = 4
5	8.0008	13.0135	18.3855	23.2571	19.1007	28.3889	36.9056	46.0399
10	9.2611	14.9056	20.4606	26.2155	20.6738	30.1288	38.9679	47.9786
15	10.0509	16.1036	21.8308	27.6911	21.5109	31.0770	39.8888	48.8832
20	10.6564	16.8714	22.6485	28.7153	22.1979	31.5292	40.4831	49.6145
25	11.1498	17.4158	23.2069	29.3030	22.8321	32.4587	40.5664	49.9824
30	11.6503	18.0000	24.1373	29.8539	23.1873	32.5142	41.2920	50.0419
35	11.8094	18.1355	24.3513	30.6235	23.5288	32.7809	41.6564	50.6674
40	12.0888	18.7087	24.6576	30.8479	23.5695	32.8632	41.9019	51.1241
50	12.5612	19.0321	25.3906	31.6847	23.9586	33.1487	42.1815	51.7205

3.2 Empirical Power Comparison

The goodness of the proposed statistic, which translates to its recommendation for use or otherwise in practical situations, is dependent on its ease of application and, more importantly, on its relative power performance. Its relative power performance is determined in this paper by comparing its power performance with those of well-known statistics for assessing whether or not a dataset is from the gamma distribution. Precisely, its power is compared with the Kolmogorov-Smirnov test, the Anderson-Darling statistics and the Henze-Meintanis-Ebner statistic. They are described in what follows:

The Kolmogorov-Smirnov KS_n Test: For each X_i observation from a random sample, obtain a rescaled observation as Y_i, where $Y_i = X_i / \hat{\beta}$. Now, let Z_i be the empirical distribution function of the Y_i and let all the $Z_i; i = 1, 2, \ldots, n$ be used to form order statistics such that $Z_{i:n}$ is the *i*th order statistic. Then, the Kolmogorov-Smirnov statistic is defined by:

$$KS_n = \max\left[\max_{1\le i\le n}\left(i/n - Z_{i:n}\right), \max_{1\le i\le n}\left(Z_{i:n} - (i-1)/n\right)\right] = \max\left(KS_n^+, KS_n^-\right)$$

Where $\max_{1\le i\le n}\left(i/n - Z_{i:n}\right) = KS_n^+$ and $\max_{1\le i\le n}\left(Z_{i:n} - (i-1)/n\right) = KS_n^-$. The test rejects the null hypothesis of gamma distribution for large values of the statistic.

The Anderson-Darling AD_n Test: Using the same definition of Z_i in the Kolmogorov-Smirnov test, the Anderson-Darling test is defined by:

$$AD_n = -n - 2\sum_{i=1}^{n}\left[\frac{2i-1}{2n}\ln Z_i + \left(1 + \frac{2i-1}{2n}\right)\ln\left(1 - Z_i\right)\right]$$

The test rejects the null hypothesis of gamma distribution for large values of the statistic.

The Henze-Meintanis-Ebner HME_n^1 and HME_n^2 Tests: Upon straightforward integration of functionals which are based on weighted integrals of the squared distance between theoretical and empirical Laplace transform of the gamma distribution, Henze, Meintanis and Ebner introduced two statistics for testing the gamma distribution. They are:

$$HME_n^1 = \frac{1}{n}\sum_{j,k=1}^{n}\left[\frac{Y_jY_k - \hat{\alpha}_n(Y_j+Y_k) + \hat{\alpha}_n^2}{Y_j+Y_k+a} + \frac{2Y_jY_k - \hat{\alpha}_n(Y_j+Y_k)}{(Y_j+Y_k+a)^2} + \frac{2Y_jY_k}{(Y_j+Y_k+a)^3}\right]$$

and

$$HME_n^2 = \frac{1}{2n}\sqrt{\frac{\pi}{a}}\sum_{j,k=1}^{n}\left[Y_jY_k + \hat{\alpha}_n^2 - \hat{\alpha}_n(Y_j+Y_k)\right]\varphi_{jk}(a)$$

$$+\frac{1}{2n}\frac{1}{2a}\sum_{j,k=1}^{n}\left[2Y_jY_k - \hat{\alpha}_n(Y_j+Y_k)\right]\left[2 - \sqrt{\frac{\pi}{a}}(Y_j+Y_k)\varphi_{jk}(a)\right]$$

$$+\frac{1}{2n}\frac{1}{4a^2}\sum_{j,k=1}^{n}Y_jY_k\left[\left\{\sqrt{\frac{\pi}{a}}(Y_j+Y_k)^2 + 2\sqrt{\pi a}\right\}\varphi_{jk}(a) - 2(Y_j+Y_k)\right]$$

Where $\varphi_{jk}(a)=\left[1-\phi\left(\frac{Y_j+Y_k}{2\sqrt{a}}\right)\right]\exp\left\{\frac{(Y_j+Y_k)^2}{4a}\right\}$ and $\phi(x)=\frac{2}{\sqrt{n}}\int_0^\infty \exp(-t^2)dt$ denotes the error function. The two tests reject the null hypothesis of the gamma distribution with large values of the statistics. In this paper, only the HME_n^1 will be used in the power comparison.

In order to carry out the empirical power comparison in this sub-section, a total of 10,000 samples are simulated in each case of sample size, *n* = 10, 20 and 50, and shape parameter, α = 1 for alternative distribution and the statistics being compared are evaluated in each case. The power of each statistic is therefore obtained as the number of the 10,000 statistics that are rejected. It is usually expressed in percentage. The alternative distributions considered in this study include the gamma distribution, $G(1,1)$; standard lognormal distribution, *LN*(0,1); the two-parameter Weibull distribution, $W(4,1)$; the standard exponential distribution; the uniform distribution, $U(0,1)$ and the beta distribution $B(1,1)$. The powers of the competing statistics at 5 percent level of significance, in percentages, are presented in Table 2.

Table 2. Empirical power comparison of tests for the gamma distribution, alpha = 0.05

Distribution	*n*	KS_n	AD_n	HME_n^1	D_n
Gamma(1,1)	10	1.9	4.9	4.8	5.0
Gamma(1,1)	20	5.7	3.4	3.8	5.2
Gamma(1,1)	50	4.9	5.4	3.9	5.1
Lognormal(0,1)	10	15.5	16.1	14.1	22.9
Lognormal (0,1)	20	18.7	22.8	33.5	31.2
Lognormal (0,1)	50	35.7	48.6	45.7	61.8
Weibull(4,1)	10	1.7	7.3	18.7	0.0
Weibull (4,1)	20	11.0	23.1	41.2	39.5
Weibull (4,1)	50	23.5	29.9	47.0	44.6
Exponential(1)	10	4.6	7.0	21.4	5.2
Exponential(1)	20	2.8	8.4	6.4	4.9
Exponential(1)	50	4.3	1.5	2.3	5.2
Uniform(0,1)	10	18.2	19.5	36.3	0.9
Uniform (0,1)	20	33.0	49.7	38.3	48.2
Uniform (0,1)	50	64.8	86.6	83.0	85.8
Beta(2,1)	10	17.6	21.0	20.9	0.8
Beta(2,1)	20	27.4	63.5	63.4	62.8
Beta(2,1)	50	66.3	87.1	87.4	88.0

From Table 2, it is clear that the D_n statistic maintained a perfect control over type-I-error, not only in its two-parameter case but also in the standard exponential distribution. This is because all its power values in both the two-parameter gamma distribution (Gamma(1,1)) and the standard exponential distribution are perfectly approximated to 5 percent, which is the level of significance. This is a desirable property of a goodness-of-fit test to a known distribution to avoid inflating the seeming power of the test with "noise". When compared with the three alternative powerful tests, there are varying degrees of fluctuations in the powers of the alternative tests at the null distribution of gamma. The variations became even more visible with the exponential distribution, which belongs to the family of gamma distribution.

Comparing the powers of the D_n statistic with the KS_n and AD_n, which are based on the distribution function, a variant of the quantile function, the results show that the proposed statistic is highly competitive with them in all the alternative distributions considered. Also, the proposed statistic, when compared with the HME_n^1 statistic, shows that it is highly competitive in all the alternative distributions considered and in all the sample sizes. It equally shows a high tendency of very high power performance as the sample size becomes very large.

4. Conclusion

In this study, a goodness-of-fit for the two-parameter gamma distribution has been developed. The statistic is seen to have good control over type-I-error. In addition, it has a relatively good power performance in relation to some time-honoured tests for the gamma distribution. Again, it is highly amenable to computer-based computations as it is easily implementable in statistical packages like the R. It is therefore recommended as a good test for the two-parameter gamma distribution.

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