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| RESEARCH ARTICLE

The Historical Background of a Famous Indeterminate Problem and Some Teaching Perspectives

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| ABSTRACT
The well-known from the History of Mathematics "hundred fowls problem" is the topic of this paper. Our main aim is to analyze the historical background of the problem and to present some ways of solving it, which have different characteristics, and each one represents a particular strategy. The interaction of Eastern and Western mathematics and the combination of different fields like Algebra and Technology in order to solve the problem has been highlighted. The study of the specific topic revealed some teaching perspectives, which we note at the end together with some concluding remarks. This paper might have implications in future research in the field of indeterminate analysis but also in Mathematics Education.

| KEYWORDS

The hundred fowls problem, Indeterminate Analysis, Diophantine equations, History of Mathematics, Mathematics Education

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1. Introduction

In the general case, solving a system of equations requires as many equations as there are unknowns. If the number of equations in the system is greater than the number of unknowns, then multiple solutions arise. Such equations are called *indeterminate*, with further constraints that unknowns are natural numbers or integers, etc. In the third century A.D., Diophantus of Alexandria attempted a systematic study, and in fact, nowadays, indeterminate equations are often called *Diophantine equations* (Keng, 1982, p. 276), although Diophantus did not deal with all possible forms, probably because he viewed some of them too easy (Millman et al., 2015, p. 62). Thus the field of Mathematics which appears in literature as *Indeterminate Analysis* is also called *Diophantine Analysis*.

Especially, if for the equation $ax + by = c$, where $a, b, c \in \mathbb{Z}$ (with a and b, not both zero) we look for integer solutions, that is, pairs of integers that verify it, then we say that we have to solve a *linear Diophantine equation*. The equation has a solution if and only if the greatest common divisor of a and b divides c. In case which $\gcd(a, b) = 1$, then the equation has infinite solutions given by the formulas

$$x = x_0 + bt \quad and \quad y = y_0 - at, \qquad t \in \mathbb{Z},$$

where (x_0, y_0) is a particular solution.

The ancient Greeks were able to solve linear indeterminate equations in the frame of *Logistic*, which also includes the treatment of the indeterminate problems of the first as well as of higher degrees (Christianidis, 1994, p. 245). Logistics is a computational aspect of Mathematics (see Bunt et al., 1988, p. 75), in contrast with *Arithmetic* in which number theory is studied.

Solving indeterminate equations nowadays is a popular subject in recreational Mathematics and math competitions at all levels, all over the world. Despite its popularity, however, how to solve indeterminate equations is rarely discussed in school classrooms. As a result, many students are lack of necessary knowledge and skills to tackle such problems (Zhou, 2015, p. 1). But even when students face indeterminate problems, they do not know what it is about. The most typical example is the "Pythagorean triplets",

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which constitute a specific category of problems in the indeterminate analysis and at the same time a part of the curriculum in high school, having many applications in modern Mathematics.

Calendar problems led Chinese mathematicians during late antiquity and the middle ages to solve systems of indeterminate linear equations, but there is no evidence for the method of solving these problems (Katz, 2009). The oldest and simplest example is found in *Sunzi suanjing* (*Sun's textbook*), written around 300 A.D.: «*There are certain things whose number is unknown. If we count them by threes, we have two left over; by fives, we have three left over; and by sevens, two are left over. How many things are there?*» (Dence & Dence, 1999, p. 156). In modern notation, the problem is to find a natural number *N* such that:

$$N \equiv 2(mod 3), \qquad N \equiv 3(mod 5), \qquad N \equiv 2(mod 7).$$

Such problems seem to have naturally led to the so-called "Chinese Remainder Theorem" (see Libbrecht, 1973, p. 215; Apostol, 1976, p. 117).

About two centuries after *Sunzi suanjing*, the book *Zhang Quijian suanjing* (*Zhang Quijian's mathematical textbook*) was written, in which, in addition to verbal formulation and the solution of indeterminate equations, the so-called "hundred fowls problem" appears for the first time, according to the traditional historiography. The original wording was something like this (Libbrecht, 1973, p. 277):

«*A cock is worth 5 ch'ien,*[1] *a hen 3 ch'ien, and 3 chicks 1 ch'ien. With 100 ch'ien we buy 100 of them. How many cocks, hens, and chicks are there?*».

This is an important problem because it occurs with different contexts and multiple variations on mathematical works in India, the Islamic world and Europe (Joseph, 2011, p. 288; Katz, 2009, p. 223) and is very useful to the historian of Mathematics (Singmaster, 2021, p. 4). Despite its importance, however, as far as we know, there are no published papers that combine the full historical analysis of the hundred fowls problem with prepositions for the mathematical elaboration (see although Baralis, 2017). It seems that the concept of indeterminate problems and their multiple valid solutions is not very common in the academic community.

The aim of this paper is to give a complete historical context of the hundred fowls problem and to analyze different approaches in the mathematical elaboration, in order to be easily accessible and usable to students, prospective teachers and math teachers as well all over the world.

After this introduction, in the second paragraph of the paper, we follow the historical journey of the hundred fowls problem in space and time, and we meet the personalities who dealt with it. Due to the nature of the problem, which requires all three species of fowls to exist, only three sets of solutions emerge. How these sets can be extracted will be dealt with within the third paragraph. Finally, in the fourth paragraph, we discuss the pedagogical perspectives and social implications of the problem.

2. Historical background

Most problems in Book II of Diophantu's *Arithmetica*, are problems of indeterminate analysis. In this book, the author illustrates some of his general methods completely by using choice and well-studied classification of the problems. The "hundred fowls problem", even though it is absent from *Arithmetica*, is, after all, a special case of a linear Diophantine equation. However, there are multiple indications that problems such as the above and the way of approaching them were already known to ancient Greeks (see Christianidis, 1994).

Zhang Quijian's suanjing last problem is the famous "hundred fowls" problem, which is often quoted by historians as one of the earliest examples involving equations with indeterminate solutions (Yong, 1997, p. 235). Another problem, prior to this type, is problem 8.13 of *Jiu zhang suanshu*, which involves five equations in six unknowns (Yong, 1994, p. 37). By using modern symbolism, the hundred fowls problem is expressed by the system

$$\begin{cases} x + y + z = 100 \\ 5x + 3y + \frac{1}{3}z = 100 \end{cases}$$

When x, y, z are the cocks, hens and chicks accordingly. Zhang Quijian himself gives three solutions: (4, 18, 78), (8, 11, 81), (12, 4, 84); however, it is not clear what method he used. In the 13th century Yang Hui, another Chinese mathematician, approached a similar problem in his work *Yang Hui Suanfa* (*Yang Hui's calculating methods*), giving a specific explanation for his methods (Yong, 1977).

A lot of years after the Chinese mathematicians who dealt with the hundred fowls problem, we meet, in England in the 8th century A.D., Charlemagne's teacher, Alcuin of York. In *Propositiones ad acuendos juvenes*, which is attributed to him by most researchers,

[1] Ch'ien is a copper coin.

he quotes 53 problems under the form of simple stories and the main aim being the pleasure from solving them (Hadley & Singmaster, 1992). Some of them are reformulations of the hundred fowls problem.

A few decades after Alcuin, about 850 A.D., a similar but more complex "hundred fowls" problem appears in the work of the Indian mathematician Mahavira, *Ganita-sarah-sangraha* (*Compendium of the Essence of Mathematics*), a wonderful sanskrit collection of arithmetical rules and problems, divided into nine chapters (Gupta, 2008, pp. 1267-1268). In this book, there is a short description of an algorithm for finding a solution to the hundred fowls problem, depending on the choice of the appropriate factors. By using modern terms, the problem can be expressed as a system of two Diophantine equations with four unknowns, which has 16 valid solutions. We do not know if Mahavira was influenced by a Chinese mathematician, although there is some evidence of an interaction between Indian and Chinese Mathematics. According to some Chinese records, there have been a number of Indian embassies in China since the 4th century, as well as Chinese visits to India (Kaye, 2020, p. 44).

Then, in the 9th-10th century, Abu Kamil, an Egyptian mathematician, writes *Kitab al-tair* (*The book of birds*), a short disquisition consisting of an introduction and six problems similar to the "hundred fowls" one (Sesiano, 2008). One of them is the following: *«With 100 dirham*[2] *we buy birds of four different types: geese for 4 dirham each, chickens for 1 dirham each, pigeons for I dirham per pair and starlings for 1 dirham per ten».* This problem is similar to Mahavira's (*Ganita-sarah-sangraha*, chapter VI, problem 152) in the sense that it can also be expressed as a system of two linear Diophantine equations with four unknowns; however, this one has 98 possible solutions. Abu Kamil's approach was unique, being the first one who concentrated on the fact that a problem like that can have so many solutions, and as he says in the introduction of his work, that was the reason he wrote *Kitabal-tair* (Sesiano, 2009).

At the beginning of the 13th century, Leonardo of Pisa wrote the masterpiece *Liber Abaci*, which contained not only computational rules with the new, for the time, Hindu-Arabic numerals but also a variety of practical problems originating mainly from the Islamic world to which Leonardo travelled many times. The last three problems of Chapter 11 are variations on the hundred fowls problem. Thus Leonardo offers the general public the well-known problem and at the same time creates a class of problems that today go under the name "recreational mathematics" (Devlin, 2011, p. 69). In *Liber Abaci,* we see a series of problems similar, and in some cases exactly identical, to the hundred fowls problem. Although Leonardo's solution is not elegant or formal, the way he expresses his thought process provides something very interesting and new. As he solves the problem, he symbolizes each step and results as a number of tables with products in them, depending on the arithmetic equation on each step (Sigler, 2002, pp. 256-257). In Leonardo's recreational mathematics, a merchant's perspective can be observed, which was quite expected because of his interest in arithmetic and in expressing mathematical problems as everyday situations, and in the case of the hundred fowls problem, as the problems of a customer or a merchant.

In 1427, a Persian mathematician and astronomer, Jamshid al-Kashi wrote in Arabic his most significant book, *Miftah al-Hisab* (*Key to Arithmetic*), in which one more variation of the hundred fowls problem appears. Obviously, the referred problem has got constant attention and a great interest in the Arabic world (Kangshen et al., 1999, pp. 420-421).

In 1525, the German mathematician Christoff Rudolff wrote the first concentrated German algebra. In his work entitled *Coss*, Rudolff displays algebraic rules (he also introduces the modern symbol for the square root) and gives multiple examples –one of them being the hundred fowls problem– that can be solved by the use of these rules.

In 1683, a Japanese mathematician, Seki Takakazu, presented a method of solving the hundred fowls problem, and other indeterminate problems in general, that is very formal, without using approximations or trials (Campbell & Higgins, 2019).

Lastly, Euler (1828, p. 313) uses a permutation of the hundred fowls problem as an introduction to the second chapter of his work problem *Elements of Algebra*, in which he develops a general method known as "*Regula Caeci*" (*blind man's rule*) commonly known as "*The rule of false position*" of solving systems of equations with three or more unknowns.

3. Mathematical elaboration

Along with the various formulations of the hundred fowls problem, many ways of solving it appeared over the centuries. One of the first methods, if not the initial, was the simple method of "trial and error" in Chinese Mathematics (Brandenburg & Nevenzeel, 2007, p. 77). Another approach, which is also due to Chinese mathematicians, is to substitute a variable with a parameter (possibly multiplied by a constant) and solve the resulting 2x2 linear system. Probably this method led Zhang Quijian to the three solutions (4, 18, 78), (8, 11, 81), (12, 4, 84), although as we saw, he does not reveal his strategy in *Zhang Quijian suanjing*.

The following approach relies more on intuition. With a series of arithmetic operations and arbitrary(?) choices of variables, Alcuin of York comes up with a unique solution. This method seems to work "in reverse", that is, having in mind a solution, obviously coming from trials, we make an effort to verify it.

[2] Dirham was a silver coin and still is a unit of currency in several states. Inherited its name from the ancient Greek currency "drachma".

Leonardo of Pisa analyzes the hundred fowls problem in simpler relationships. He finds how many and which fowls one can buy, e.g. with 4 or 5 coins and then creates a linear combination of the above relationships to achieve the given fowl numbers and coins. The solution is described verbally, while in the margins of the pages of *Liber Abaci* the intermediate relations are noted.

In a more complex case of a Diophantine system of two equations with four unknowns, the usual practice dictates the deletion of one variable in order to obtain a new equation with three variables. The next two variables are replaced by equal numbers of parameters multiplied by certain coefficients, so that all terms have common divisors. Afterwards, in a table are listed all the possible values of the parameters, while then some are deleted with the current restrictions. All that remains is the accepted integer solutions. This method is essentially the way Mahavira and Abu Kamil worked.

Here are four possible ways to solve the hundred fowls problem, based on elementary mathematics or using widespread Information and Communication Technology tools like the dynamic mathematics software "GeoGebra".

1st way

Let $x, y, z \in \mathbb{N}$ $(x < 20, y < 33, z < 300)$ the number of cocks, hens and chicks, respectively. Then we have:

$$\begin{cases} x + y + z = 100 \\ 5x + 3y + \frac{1}{3}z = 100 \end{cases} \Leftrightarrow \begin{cases} z = 100 - x - y \\ 15x + 9y + z = 300 \end{cases}$$

The first equation is solved for z, so we substitute the expression $100 - x - y$ in for z in the second equation:

$$15x + 9y + 100 - x - y = 300 \Leftrightarrow 14x + 8y = 200 \Leftrightarrow 7x + 4y = 100 \Leftrightarrow$$

$$y = -\frac{7}{4}x + 25$$

The last equation indicates that x must be divisible by 4, so:

- If $x = 4$, then $y = 18$ and $z = 78$
- If $x = 8$, then $y = 11$ and $z = 81$
- If $x = 12$, then $y = 4$ and $z = 84$
- If $x = 16$, then $y = -3$ (this case is rejected)

Therefore, the three valid solutions are:

$$(x, y, z) = (4, 18, 78) \ or \ (8,11,81) \ or \ (12,4,84).$$

2nd way

Let $x, y, z \in \mathbb{N}$ $(x < 20, y < 33, z < 300)$ the number of cocks, hens and chicks, respectively. In the following system

$$\begin{cases} x + y + z = 100 \\ 5x + 3y + \frac{1}{3}z = 100 \end{cases}$$

We pose $z = 3t$, where t is a positive parameter. So we take the system:

$$\begin{cases} x + y + 3t = 100 \\ 5x + 3y + t = 100 \end{cases} \Leftrightarrow \begin{cases} x + y = 100 - 3t \\ 5x + 3y = 100 - t \end{cases}$$

and we solve it by determinants. First, we find the determinant D of the matrix of coefficients:

$$D = \begin{vmatrix} 1 & 1 \\ 5 & 3 \end{vmatrix} = 3 - 5 = -2$$

Since $D \neq 0$ the system has a unique solution. To obtain the numerators D_x and D_y we replace in the matrix of coefficients, the coefficients of x and y respectively by the constant terms:

$$D_x = \begin{vmatrix} 100 - 3t & 1 \\ 100 - t & 3 \end{vmatrix} = 300 - 9t - 100 + t = -8t + 200$$

$$D_y = \begin{vmatrix} 1 & 100 - 3t \\ 5 & 100 - t \end{vmatrix} = 100 - t - 500 + 15t = 14t - 400$$

Then the unique solution of the system is

$$x = \frac{D_x}{D} = \frac{-8t + 200}{-2} = 4t - 100$$

$$y = \frac{D_y}{D} = \frac{14t - 400}{-2} = -7t + 200$$

But there are the following restrictions:

$$0 < x < 20 \ \ and \ \ 0 < y < 33$$

$$0 < 4t - 100 < 20 \ \ and \ \ 0 < -7t + 200 < 33$$

$$25 < t < 30 \ \ and \ \ \frac{167}{7} < t < \frac{200}{7}$$

Taking into account the values of t intersect with one another, we have:

- If $t = 26$, then $x = 4, y = 18, z = 78$
- If $t = 27$, then $x = 8, y = 11, z = 81$
- If $t = 28$, then $x = 12, y = 4, z = 84$

Therefore, the three valid solutions are: $(x, y, z) = (4, 18, 78)\ or\ (8,11,81)\ or\ (12,4,84)$.

3rd way

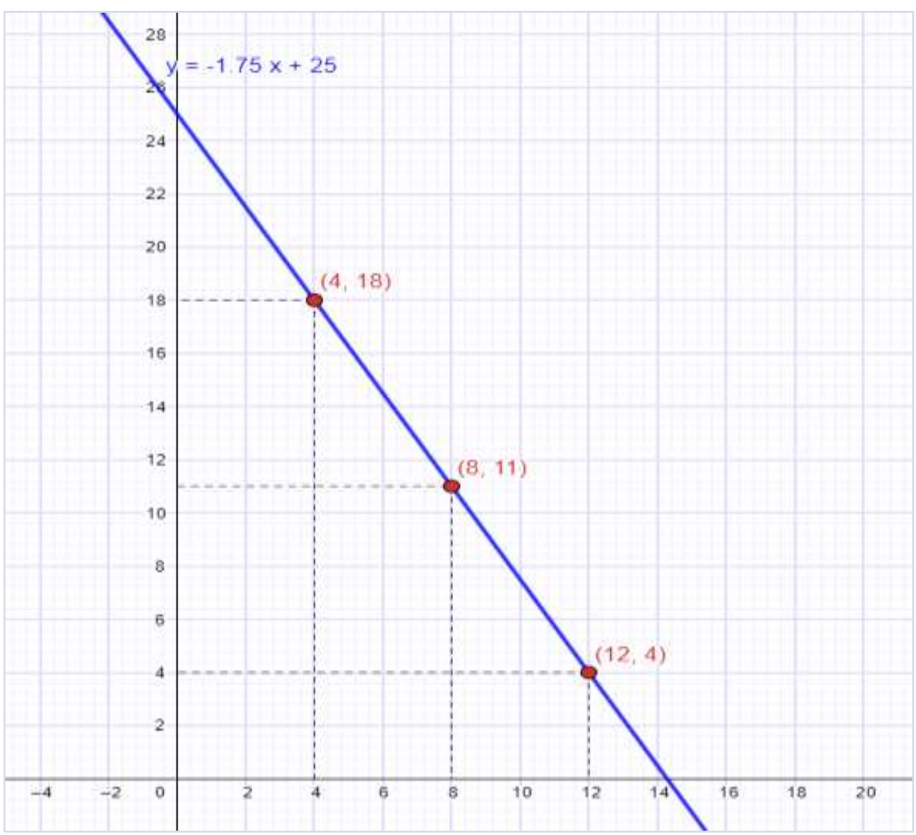
The third way begins as the first way but concentrates on the integration of dynamic geometry software. A similar approach, i.e. the incorporation of the dynamic mathematics software into History of Mathematics, has been attempted by other researchers with positive results (see Zengin, 2018; Meadows & Caniglia, 2021).

With the help of GeoGebra, we draw in the plane $\mathbb{R}^2$ the line with equation

$$y = -\frac{7}{4}x + 25$$

with the additional conditions that $x > 0$ and $y > 0$.

Figure 1: The graph of the line $y = -\frac{7}{4}x + 25$



The problem data indicates that we focus only on the first quadrant, and we take into account only points on the line with integer coordinates.

So we get:

$$(x, y) = (4, 18) \text{ } or \text{ } (x, y) = (8, 11) \text{ } or \text{ } (x, y) = (12, 4).$$

However, it should be borne in mind that significant difficulties arise in the transition from the intuitive to the theoretical level (Fischbein, 1987). If proper pedagogical care is not taken, a simple operation of the mathematical software is capable of convincing students of the correctness of what it deals with (!). It is, therefore, necessary to clarify that *discovery* (or design) with the help of software and mathematical *proof* is two related but different things (Mariotti, 2000). In practice, this means that we have to algebraically verify the points we found by observing the above graph.

Finally, from the equation $x + y + z = 100$ we get the according values of the variable z. Therefore, the valid values of z are:

$$z = 78 \text{ } or \text{ } z = 81 \text{ } or \text{ } z = 84.$$

4th way

The hundred fowls problem, as we saw above, leads to the equation

$$y = -\frac{7}{4}x + 25$$

with the additional conditions that $x > 0$ and $y > 0$, or equivalent to

$$7x + 4y = 100$$

which is a special case of the Diophantine equation; actually is a linear Diophantine equation and can be solved based on the Euclidean algorithm. It is known that since the greater common divisor of 7 and 4 is 1, then the last equation has infinite solutions given by the formulas

$$x = x_0 + 4t \quad and \quad y = y_0 - 7t, \qquad t \in \mathbb{Z},$$

where (x_0, y_0) is a particular solution. We observe that an (obvious) solution is

$$(x_0, y_0) = (4,18),$$

so

$$x = 4 + 4t \quad and \quad y = 18 - 7t, \qquad t \in \mathbb{Z},$$

and we create the following table (Table 1). To create the table, we can use a spreadsheet application (e.g. MsExcel) or a similar program:

Table 1: Shows the values of the variables x and y, for the various values of the parameter t

t	x	y
...	...	...
-2	-4	32
-1	0	25
0	4	18
1	8	11
2	12	4
3	16	-3
4	20	-10

After putting the formulas at the table, we observe that the only valid solutions are the following pairs:

$$(x, y) = (4, 18) \text{ } or \text{ } (x, y) = (8, 11) \text{ } or \text{ } (x, y) = (12, 4)$$

Then, from the equation:

$$x + y + z = 100$$

We get the values of the variable z. Therefore:

$$z = 78 \; or \; z = 81 \; or \; z = 84$$

respectively. Historically, Brahmagupta seems to be closer to this method, although it remains questionable whether Indian mathematicians learned the Euclidean algorithm from the ancient Greeks (Katz, 2009, pp. 244-247).

4. Teaching perspectives

In the previous paragraph, we mentioned four indicative ways to solve the hundred fowls problem, which could be utilized by math teachers, according to their judgment and in relation to the educational aims and objectives they set for their classes. We consider, depending on the curriculum of each country, that the hundred fowls problem could possibly be taught in high school (10^{th}-12^{th} grade) or even in the first years of tertiary education. For example, in Greece, as in most European countries, this subject, although it is not part of the syllabus, can be taught in an Algebra class in the 11^{th} grade or in the first year in University in a Number Theory or History of Mathematics course. But both students and prospective teachers can deal with this "recreational problem" since such problems have great pedagogic utility and are often the basis of serious Mathematics (Singmaster, 2021).

The above approach combines mathematical with historical components by taking advantage of a compulsory mathematical course in school (e.g. Algebra) or in college together with elements from the History of Mathematics, namely the well-known "hundred fowls problem" offered to students. In this way, the History of Mathematics can be integrated into the teaching and learning of Mathematics (see Fried, 2001; Furinghetti, 2020) as a *tool*, that is, a means of learning the content of Mathematics (see Tzanakis & Arcavi, 2000; Jankvist, 2009).

The hundred fowls problem, in addition to the very important historical basis it sets, is essentially an "open-ended" and at the same time "open-minded" problem, in the sense that it is open to several correct answers as well as various ways of solving (see Becker & Shimada, 1997). Such problems contribute to students' involvement in a productive inquiry (Schoenfeld, 1985) and enable them to accept the existence of different but equally correct solutions in the process of the quest for knowledge and understanding. It would therefore be interesting to conduct future research investigating whether the historical-cultural context of a problem is capable of involving students in the educational process, as well as how students respond to open-ended problems.

However, the question of whether we indeed wish to pose problems like the "hundred fowls" remains open and crucial since such problems are missing from everyday school and university practice. We consider that, especially for university students, this –at least as we comprehend it– would be a realistic "connection to the labor market", that is, the young scientists' cultivation of their *adaptability* to situations that are changing and not the consolidation of *uncertainty* in working life.

5. Conclusion

This paper aimed to give a complete historical context of the hundred fowls problem and to analyze different approaches in the mathematical elaboration in order to be easily accessible and usable to students, prospective teachers and math teachers. As highlighted by the historical background (second paragraph), the hundred fowls problem is a puzzle that has permeated the mathematical tradition of both Eastern and Western civilizations for hundreds of years and could play the role of a symbol that highlights the age-old cultural relations between civilizations (Hermelink, 1978). At the same time, with the plethora of ways to solve it (these ways are set out in the third paragraph), it underlines the crucial role of Mathematics through its interaction with other fields and aspects such as History and Technology. Apart from all this, however, it seemed to have a special interest in Mathematics Education, an interest which future research could explore in-depth. For example, a teaching experiment could be conducted in a class that would combine History of Mathematics with *problem-solving processes* (see, e.g. Polya, 1981), focusing on an open-ended problem such as that of a hundred fowls. Thus, this article could contribute to the education of pupils, Mathematics' students, prospective teachers and Math teachers, as well as to the research concerning the utilization of problem-solving processes and the integration of the History of Mathematics in teaching and learning Mathematics.

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