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| RESEARCH ARTICLE

Optical Solitons in Fiber Bragg Gratings for Fractional Nonlinear Schrödinger Equation with Generalized Anti-cubic Nonlinearity using Conformable Derivative

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| ABSTRACT
This work explores Kink soliton solution, periodic soliton solution, and rational function solutions for the fractional generalized anti-cubic (FGAC) nonlinearity in fiber Bragg gratings (BGs). The rational fractional $(\frac{D_{\xi}^{\alpha}G}{G})$-expansion method is employed in conjunction with the idea of conformable fractional derivative. Due to its nature, the soliton solution looks to have some restrictions.

| KEYWORDS

Solitons, fractional generalized anti-cubic nonlinearity, Bragg gratings, the rational fractional $(\frac{D_{\xi}^{\alpha}G}{G})$-expansion method.

| ARTICLE INFORMATION

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1. Introduction
Optical soliton in fiber Bragg gratings is one of the fascinating areas of research in optoelectronics [Zayed, et al. 2020]. Despite the fact that most publications focus on the integrability features of the equations, the fractional complex-valued function has been progressively prevalent. Fiber Bragg gratings are excellent sensor components that can be used to measure a variety of engineering parameters, such as tilt, pressure, strain, temperature, acceleration, displacement, load, and even the presence of variety of manufacturing, biomedical and chemical matter, using both static and dynamic mechanisms. Differential group delay arises if an optical fiber's rotational symmetry is disrupted. Mechanical tension, as well as random flaws or bending of fibers, could induce this effect. Finally, temperature variations might have an impact on optical fiber homogeneity. The phenomenon of birefringence is brought to light by the cumulative effect of differential group delay across transcontinental distances. This causes pulse splitting in optical fiber, and the governing equations become vector-coupled equations. A wide variety of techniques are available, including the symmetry method [Kumar, et al. 2019], the rational $(\frac{G'}{G})$-expansion method [Akbar, 2019], the generalized $(\frac{G'}{G})$-expansion method [Foroutan, 2018], the rational fractional $(\frac{D_{\xi}^{\alpha}G}{G})$-expansion method [Atwaty, 2021] have been formulated to achieve exact solutions of NLSE.

In this article, we further investigate the fractional generalized anti-cubic nonlinearity in fiber Bragg gratings that were introduced for the first time in [Biswas, 2019]. The idea of the conformable fractional derivative will be used. After that, the rational fractional $(\frac{D_{\xi}^{\alpha}G}{G})$-expansion method shall be carried out for the first time in order to find optical soliton solutions for the two models.

2. Governing Model

The FNLSE with gAC nonlieearity [Biswas, 2019]

$$iD_t^{\alpha} q + aD_x^{2\beta} q + (\frac{b_1}{|q|^{2n+2}} + b_2|q|^{2n} + b_3|q|^{2n+2})q = 0. \tag{1}$$

The dynamic mechanism that characterizes the propagation of pulses through the fibers is denoted by $q(x,t)$. The linear temporal evolution is depicted in the first notion, whereas the chromatic dispersion (CD) is reflected by the factor a, the coefficients b_j for $j = 1,2$ stands for self-phase modulation (SPM). the power-law nonlinearity assigned by n where $-1 < n < 3$ [Biswas, 2019]. Generalized gAC nonlinearity in fiber BGs with two cases will be discussed in the following subsections. For $n = n$ on the usual knowing equation and for $n = 0$ on the vector-coupled model corresponding to gAC nonlinearity.

CASE-I: The FNLSE with gAC nonlieearity for $\boldsymbol{n} = \boldsymbol{0}$ [16];

$$\begin{aligned} &iD_t^{\alpha}\psi + a_1 D_x^{2\beta}\phi + \frac{b_1\psi}{c_1|\psi|^2 + d_1|\phi|^2} + e_1\psi + (f_1|\psi|^2 + g_1|\phi|^2)\psi + ih_1 D_x^{\beta}\psi + k_1\phi = 0, \\ &iD_t^{\alpha}\phi + a_2 D_x^{2\beta}\psi + \frac{b_2\phi}{c_2|\phi|^2 + d_2|\psi|^2} + e_2\phi + (f_2|\phi|^2 + g_2|\psi|^2)\phi + ih_2 D_x^{\beta}\phi + k_2\psi = 0, \end{aligned} \tag{2}$$

where $a_l, b_l, c_l, d_l, e_l, f_l, g_l, h_l,$ and k_l for $l = 1,2$ are constants. The x and t variables indicate spatial and temporal variables respectively. $\psi(x,t)$ and $\phi(x,t)$ are the dependent variables, which represent wave profiles along the two components. The coefficients a_l are used to express dispersive reflectivity, The coefficients b_l are represents the combination of SPM and cross-phase modulation effects XPM. The coefficients c_l, e_l and f_l define SPM, while the coefficients d_l and g_l define XPM. The inter-modal dispersion coefficient is then h_l, and finally the detuning coefficient is k_l

3. Mathematical Preliminaries

In this paper, the rational fractional $(\frac{D_\zeta^{\alpha} G}{G})$-expansion method is used to obtain analytical solutions to Eqs. (1) and (2). The approach made use of the concept of conformable fractional derivative, which has the definition as [Islam, et al. 2020].

$$\frac{d\psi}{dx} = \lim_{\epsilon\to 0} \frac{\psi(x+\epsilon) - \psi(x)}{\epsilon},$$

where $\psi(x)\colon [0,\infty] \to \mathbb{R}$ and $x > 0$. For the traditional definition, we have $\frac{d(x^n)}{dx} = nx^{n-1}$. Accordingly, Khalil introduced α order fractional derivative of ψ as

$$T_\alpha \psi(x) = \lim_{\epsilon\to 0} \frac{\psi(x + \epsilon x^{1-\alpha}) - \psi(x)}{\epsilon}, \quad 0 < \alpha \le 1,$$

for $\psi(x)$ is α differentiable and $\lim_{x\to 0^+} T_\alpha \psi(x)$ exists, then the conformable derivative at $x = 0$ is defined as $T_\alpha \psi(0) = \lim_{x\to 0^+} T_\alpha \psi(x)$.

If $u(x)$ and $v(x)$ are α-differentiable at any point $x > 0$, for $\alpha \in (0,1]$, then [Islam, et al. 2020]:

$$T_\alpha(au + bv) = aT_\alpha(u) + bT_\alpha(v) \quad \forall a, b \in \mathbb{R}. \tag{3}$$

$$T_\alpha(x^n) = nx^{n-\alpha} \quad \forall n \in \mathbb{R}. \tag{4}$$

$$T_c = 0, \; where \; c \; is \; a \; constant\,. \tag{5}$$

$$T_\alpha(uv) = uT_\alpha(v) + vT_\alpha(u). \tag{6}$$

$$T_\alpha = \frac{vT_\alpha(u) - uT_\alpha(v)}{v^2}. \tag{7}$$

If u is differentiable, then $\quad T_\alpha(u(x)) = x^{1-\alpha}\frac{du(x)}{dx}$. (8)

To solving the considered equation we set

$$q(x,t) = u(\zeta(x,t))e^{i\theta(x,t)}, \tag{9}$$

where ζ and θ denotes the soliton amplitude component and the soliton phase component respectively, which are characterized as

$$\zeta(x,t) = x - \lambda \frac{t}{\alpha}, \tag{10}$$

$$\theta = Jx + w\frac{t}{\alpha} + k. \tag{11}$$

Substituting Eqs.(9),(10) and (11) into Eq.(1) we get

$$-(w + aJ^2)u^{2n+2} + i(2aJ - \lambda)u^{2n+1}D_\zeta^\alpha u + au^{2n+1}D_\zeta^{2\alpha}u + b_1 + b_2u^{4n+2} + b_3u^{4n+4} = 0 \tag{12}$$

Balancing $u^{2n+1}D_\zeta^{2\alpha}$ with u^{4n+4} gives $N = \frac{1}{n+1}$. Setting $u(\zeta) = [U(\zeta)]^{\frac{1}{n+1}}$ we get

$$-(w + aJ^2)U^2 + \frac{i}{n+1}(2aJ - \lambda)UD_\zeta^\alpha U + \frac{a}{n+1}UD_\zeta^{2\alpha}U - \frac{an}{(n+1)^2}(D_\zeta^\alpha U)^2 + b_1 + b_2U^{\frac{4n+2}{n+1}} + b_3U^4 = 0, \tag{13}$$

the real part of Eq.(13) is

$$-(w + aJ^2)U^2 + \frac{a}{n+1}UD_\zeta^{2\alpha}U - \frac{an}{(n+1)^2}(D_\zeta^\alpha U)^2 + b_1 + b_2U^{\frac{4n+2}{n+1}} + b_3U^4 = 0, \tag{14}$$

and the imaginary part of Eq.(13) is

$$[\frac{1}{n+1}(2aJ - \lambda)UD_\zeta^\alpha U] = 0. \tag{15}$$

From Eq.(15) we have

$$\lambda = 2aJ \tag{16}$$

For integrability, we must set $b_2 = 0$ in Eq.(14) this leads to

$$UD_\zeta^{2\alpha}U - \rho_1U^2 - \frac{n}{n+1}(D_\zeta^\alpha U)^2 + \rho_2 + \rho_3U^4 = 0, \tag{17}$$

where $\rho_1 = \frac{(n+1)(w+aJ^2)}{a}$, $\rho_2 = \frac{b_1(n+1)}{a}$, $\rho_3 = \frac{b_3(n+1)}{a}$

The rational Fractional $(\frac{D_\zeta^\alpha G}{G})$-expansion Method

Step 1: Assume Eq.(17) has a solution of the type

$$U(\zeta) = \frac{\sum_{i=0}^N A_i(\frac{D_\zeta^\alpha G}{G})^i}{\sum_{i=0}^N B_i(\frac{D_\zeta^\alpha G}{G})^i}, \tag{18}$$

where $A_i(i = 0,1,2,\ldots,N)$ and $B_i(i = 0,1,2,\ldots,N)$ undetermined parameters, and $G(\zeta)$ satisfy the following auxiliary fractional differential equation [16]

$$D_\zeta^{2\alpha}G + \eta D_\zeta^\alpha G + \mu G = 0, \tag{19}$$

It is commonly known that the following solutions exist for equation (19):

$$(\frac{D_\zeta^\alpha G}{G}) = \begin{cases} \frac{\sqrt{\eta^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta}{2}, & \eta^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu-\eta^2}}{2}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-c_2\cos\left(\frac{\sqrt{4\mu\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}\right)-\frac{\eta}{2}, & \eta^2 - 4\mu < 0, \\ \frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^\alpha}-\frac{\eta}{2}, & \eta^2 - 4\mu = 0. \end{cases} \tag{20}$$

4. Application

Balancing $UD_{\zeta}^{2\alpha}U$ with U^4 we obtain $N=1$. Inserting Eq.(18) with $N=1$ into Eq.(17), using Eq.(18) and collect the coefficients of each power of $(\frac{D_{\zeta}^{\alpha}G}{G})^i(\zeta),(i=0,1,2,3,4,5)$ we obtain a set of algebraic equations in the unknown A_0, A_1, B_0, and B_1. Solving this system using Matlab we have:

Case 1: when $B_0=\mp\frac{2A_0}{\eta}\sqrt{\frac{\rho_3(n+1)}{3n+2}},\ B_1=0,\ A_0=A_0,\ A_1=\frac{2A_0}{\eta},\ \rho_1=\frac{(2n+1)(\eta^2-4\mu)}{2(n+1)},\ \rho_2=\frac{n(3n+2)(4\mu-\eta^2)^2}{16\rho_3(n+1)^2}$. Provide $\rho_3\neq 0$ and $n\neq -1$. As a result, Eq.(1) has the following traveling wave solutions:

when $\eta^2-4\mu>0$, we have the following hyperbolic function solution

$$q(x,t)=\left[\mp\sqrt{\frac{(3n+2)(\eta^2-4\mu)}{\rho_3(n+1)}}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)\right]^{\frac{1}{n+1}}\times\ e^{i(Jx+\frac{a(2n+1)(\eta^2-4\mu)t}{2(n+1)^2-aJ^2\ \ \alpha}+k)}. \quad (21)$$

For $c_1\neq 0,\ c_2=0$ then equation (21) becomes

$$q(x,t)=\left[\mp\sqrt{\frac{(3n+2)(\eta^2-4\mu)}{\rho_3(n+1)}}\left(\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)\right)\right]^{\frac{1}{n+1}}\times e^{i(Jx+\frac{a(2n+1)(\eta^2-4\mu)t}{2(n+1)^2-aJ^2\ \ \alpha}+k)}, \quad (22)$$

when $\eta^2-4\mu<0$, we have the following trigonometric function solution

$$q(x,t)=\left[\mp\sqrt{\frac{(3n+2)(4\mu-\eta^2)}{\rho_3(n+1)}}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)\right]^{\frac{1}{n+1}}\times\ e^{i(Jx+\frac{a(2n+1)(\eta^2-4\mu)t}{2(n+1)^2-aJ^2\ \ \alpha}+k)}. \quad (23)$$

For $c_1\neq 0,\ c_2=0$ we have

$$q(x,t)=\left[\mp\sqrt{\frac{(3n+2)(4\mu-\eta^2)}{\rho_3(n+1)}}\left(\tan\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)\right)\right]^{\frac{1}{n+1}}\times\ e^{i(Jx+\frac{a(2n+1)(\eta^2-4\mu)t}{2(n+1)^2-aJ^2\ \ \alpha}+k)}, \quad (24)$$

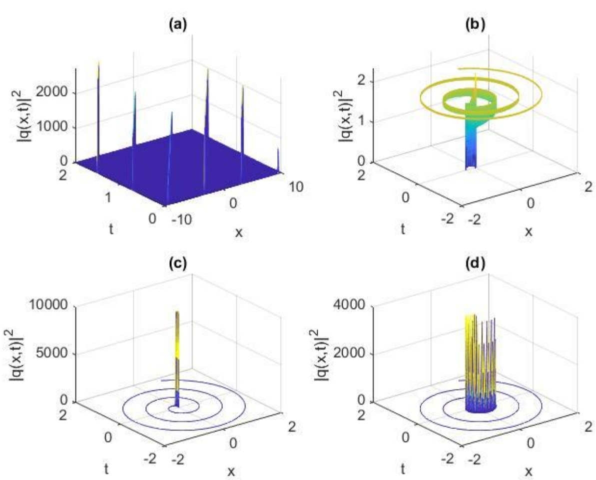

Figure 1: The solution |q(x,t)|^2 of Eq.(24) with a=2, J=2, n=1, b_3=1, η=3, μ=0.5, and x∈[-10,10], t∈[0,2], and α=1 for Figure (a) and in cylindrical coordinates α=0.25 for Figure (b), α=0.5 for Figure (c) and α=0.75 for Figure (d).

when $\eta^2-4\mu=0$, we have the following rational function solution

$$q(x,t)=\left[\mp\sqrt{\frac{(3n+2)}{\rho_3(n+1)}}\left(\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^\alpha}\right)\right]^{\frac{1}{n+1}}\times e^{i(Jx+k)}. \tag{25}$$

For $C_1=0,\ C_2\neq 0$ we have

$$q(x,t)=\left[\mp\sqrt{\frac{(3n+2)}{\rho_3(n+1)}}\left(\frac{\Gamma(1+\alpha)}{\zeta^\alpha}\right)\right]^{\frac{1}{n+1}}\times e^{i(Jx+k)}. \tag{26}$$

Case 2: when $B_0=\pm\frac{2A_0\eta}{\eta^2-4\mu}\sqrt{-\frac{\rho_3(n+1)}{(n+2)}}$, $B_1=\pm\frac{4A_0}{\eta^2-4\mu}\sqrt{-\frac{\rho_3(n+1)}{(n+2)}}$, $A_0=A_0$, $A_1=0$, $\rho_1=\frac{4\mu-\eta^2}{2(n+1)}$, $\rho_2=-\frac{n(n+2)(4\mu-\eta^2)^2}{16\rho_3(n+1)^2}$. Provide $\rho_3\neq 0$ and $n\neq -1$. As a result, Eq.(1) has the following traveling wave solutions:

when $\eta^2-4\mu>0$, we have the following hyperbolic function solution

$$q(x,t)=\left[\frac{\eta^2-4\mu}{\sqrt{\frac{\rho_3(n+1)(4\mu-\eta^2)}{(n+2)}}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)}\right]^{\frac{1}{n+1}}\times\ e^{i(Jx+(\frac{4\mu-\eta^2}{2(n+1)}-aJ^2)\frac{t}{\alpha}+k)}. \tag{27}$$

For $c_1\neq 0,\ c_2=0$ then equation (27) becomes

$$q(x,t)=\left[\frac{\eta^2-4\mu}{\sqrt{\frac{\rho_3(n+1)(4\mu-\eta^2)}{(n+2)}}\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(\frac{4\mu-\eta^2}{2(n+1)}-aJ^2)\frac{t}{\alpha}+k)}, \tag{28}$$

when $\eta^2-4\mu<0$, we have the following trigonometric function solution

$$q(x,t)=\left[\frac{\eta^2-4\mu}{\sqrt{\frac{\rho_3(n+1)(4\mu-\eta^2)}{(n+2)}}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)}\right]^{\frac{1}{n+1}}\times\ e^{i(Jx+(\frac{4\mu-\eta^2}{2(n+1)}-aJ^2)\frac{t}{\alpha}+k)}. \tag{29}$$

$$q(x,t)=\left[\frac{\eta^2-4\mu}{\sqrt{\frac{\rho_3(n+1)(4\mu-\eta^2)}{(n+2)}}\tan\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(\frac{4\mu-\eta^2}{2(n+1)}-aJ^2)\frac{t}{\alpha}+k)} \tag{30}$$

Case 3: when $B_0=0$, $B_1=\pm\frac{A_0}{\mu}\sqrt{-\frac{\rho_3(n+1)}{(n+2)}}$, $A_0=A_0$, $A_1=\frac{A_0\eta}{2\mu}$, $\rho_1=\frac{4\mu-\eta^2}{2(n+1)}$, $\rho_2=-\frac{n(n+2)(4\mu-\eta^2)^2}{16\rho_3(n+1)^2}$. Provide $\rho_3\neq 0$ and $n\neq -1$. As a result, Eq.(1) has the following traveling wave solutions:

when $\eta^2-4\mu>0$, we have the following hyperbolic function solution

$$q(x,t)=\left[\frac{2\mu+\frac{\eta\sqrt{\eta^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta}{2}}{\sqrt{\frac{\rho_3(n+1)(\eta^2-4\mu)}{4(n+2)}}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times\; e^{i(Jx+(\frac{4\mu-\eta^2}{2(n+1)}-aJ^2)\frac{t}{\alpha}+k)}. \quad (31)$$

For $c_1\neq 0,\ c_2=0$ then equation (31) becomes

$$q(x,t)=\left[\frac{2\mu+\frac{\eta\sqrt{\eta^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta}{2}}{\sqrt{\frac{\rho_3(n+1)(\eta^2-4\mu)}{4(n+2)}}\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(\frac{4\mu-\eta^2}{2(n+1)}-aJ^2)\frac{t}{\alpha}+k)}, \quad (32)$$

when $\eta^2-4\mu<0$, we have the following trigonometric function solution

$$q(x,t)=\left[\frac{2\mu+\frac{\eta\sqrt{4\mu-\eta^2}}{2}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta}{2}}{\sqrt{\frac{\rho_3(n+1)(4\mu-\eta^2)}{4(n+2)}}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times\; e^{i(Jx+(\frac{4\mu-\eta^2}{2(n+1)}-aJ^2)\frac{t}{\alpha}+k)}. \quad (33)$$

For $c_1\neq 0,\ c_2=0$ we have

$$q(x,t)=\left[\frac{2\mu+\frac{\eta\sqrt{\eta^2-4\mu}}{2}\tan\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta}{2}}{\sqrt{\frac{\rho_3(n+1)(\eta^2-4\mu)}{4(n+2)}}\tan\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(\frac{4\mu-\eta^2}{2(n+1)}-aJ^2)\frac{t}{\alpha}+k)} \quad (34)$$

when $\eta^2-4\mu=0$, we have the following rational function solution

$$q(x,t)=\left[\frac{2\mu+\eta\left(\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^\alpha}\right)-\frac{\eta}{2}}{\sqrt{\frac{\rho_3(n+1)}{(n+2)}}\left(\frac{2C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^\alpha}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx-\frac{aJ^2}{\alpha}t+k)}. \quad (35)$$

For $c_1=0,\ c_2\neq 0$ we have

$$q(x,t)=\left[\frac{2\mu+\eta\left(\frac{C_2\Gamma(1+\alpha)}{\zeta^\alpha}\right)-\frac{\eta}{2}}{\sqrt{\frac{\rho_3(n+1)}{(n+2)}}\left(\frac{\Gamma(1+\alpha)}{\zeta^\alpha}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx-\frac{aJ^2}{\alpha}t+k)}. \quad (36)$$

5. Mathematical Analysis: CASE-I (n=0)

The following is the initial hypothesis for addressing the considered coupled system:

$$\psi(x,t)=u_1(\zeta)e^{i\theta}, \quad (37)$$

$$\phi(x,t)=u_2(\zeta)e^{i\theta}. \quad (38)$$

Substituting Eqs.(37) and Eq.(38) into Eq.(2) we get

$$-i\,\lambda\,D_{\zeta}^{\alpha}u_1 - w\,u_1 - a_1J^2u_2 + 2\,i\,a_1\,J\,D_{\zeta}^{\alpha}u_2 + a_1\,D_{\zeta}^{2\alpha}u_2 + \frac{b_1u_1}{(c_1u_1^2+d_1u_2^2)} + e_1u_1$$
$$+(f_1u_1^2 + g_1u_2^2)u_1 + ih_1\,D_{\zeta}^{\alpha}u_1 - h_1\,J\,u_1 + k_1\,u_2 = 0, \tag{39}$$

$$-i\,\lambda\,D_{\zeta}^{\alpha}u_2 - w\,u_2 - a_2J^2u_1 + 2\,i\,a_2\,J\,D_{\zeta}^{\alpha}u_1 + a_2\,D_{\zeta}^{2\alpha}u_1 + \frac{b_2u_2}{(c_2u_2^2+d_2u_1^2)} + e_2u_2$$
$$+(f_2u_2^2 + g_2u_1^2)u_2 + ih_2\,D_{\zeta}^{\alpha}u_2 - h_2\,J\,u_2 + k_2\,u_1 = 0, \tag{40}$$

Eqs.(39) and (40) can be gathered as

$$-i\,\lambda\,D_{\zeta}^{\alpha}u_j - w\,u_j - a_jJ^2u_l + 2\,i\,a_j\,J\,D_{\zeta}^{\alpha}u_l + a_j\,D_{\zeta}^{2\alpha}u_l + \frac{b_ju_j}{(c_ju_j^2+d_ju_l^2)} + e_ju_j$$
$$+(f_ju_j^2 + g_ju_l^2)u_j + ih_j\,D_{\zeta}^{\alpha}u_j - h_j\,J\,u_j + k_ju_l = 0, \tag{41}$$

where $j = 1{,}2$ and $l = 3 - j$. Using the balance principle we get $u_j = u_l$ and that leads to

$$a_j\,D_{\zeta}^{2\alpha}u_j + (k_j + e_j - w - h_j\,J - a_j\,J^2)u_j + \frac{b_j}{(c_j+d_j)u_j} + (f_j + g_j)u_j^3 - i\big(\lambda - 2\,a_j\,J - h_j\big)D_{\zeta}^{\alpha}u_j = 0, \tag{42}$$

the real part of Eq.(42) is

$$u_jD_{\zeta}^{2\alpha}u_j + \rho_1u_j^2 + \rho_2 + \rho_3u_j^4 = 0, \tag{43}$$

the imaginary part of Eq.(42) is

$$\lambda - 2\,a_j\,J - h_j = 0. \tag{44}$$

Where $\rho_1 = \frac{k_j+e_j-w-h_j\,J-a_j\,J^2}{a_j}$, $\rho_2 = \frac{b_j}{a_j(c_j+d_j)}$, and $\rho_3 = \frac{(f_j+g_j)}{a_j}$. Provide that $a_j \neq 0$ and $c_j + d_j \neq 0$. From Eq.(44) we have

$$\lambda = 2\,a_j\,J + h_j. \tag{45}$$

6. Application

Balancing $u_jD_{\zeta}^{2\alpha}u_j$ with u_j^4 provides $N = 1$, which is then substituted into Eq.(18) and subsequently Eq.(43) and solve the result algebraic equations using Matlab yields:

Case 1: when $B_0 = \frac{A_0\sqrt{2\rho_3}}{\eta}$, $B_1 = 0$, $A_0 = A_0$, $A_1 = \frac{2A_0}{\eta}$, $\rho_1 = 2\mu - \frac{\eta^2}{2}$, $\rho_2 = 0$. Provide $\eta \neq 0$. As a result, Eq.(2) has the following traveling wave solutions:

when $\eta^2 - 4\mu > 0$, we have the following hyperbolic function solution

$$\psi(x,t) = \left[\mp\sqrt{\frac{\eta^2-4\mu}{2\rho_3}}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{2}\right)}\right)\right]^{\frac{1}{n+1}} \times \; e^{i(Jx+(k_1+e_1-h_1J-a_1J^2-2a_1\mu+\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \tag{46}$$

$$\phi(x,t) = \left[\mp\sqrt{\frac{\eta^2-4\mu}{2\rho_3}}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{2}\right)}\right)\right]^{\frac{1}{n+1}} \times \; e^{i(Jx+(k_2+e_2-h_2J-a_2J^2-2a_2\mu+\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \tag{47}$$

For $c_1 \neq 0$, $c_2 = 0$ we have

$$\psi(x,t) = \left[\mp\sqrt{\frac{\eta^2-4\mu}{2\rho_3}}\left(\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)\right)\right]^{\frac{1}{n+1}} \times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2-2a_1\mu+\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}, \tag{48}$$

$$\phi(x,t) = \left[\mp\sqrt{\frac{\eta^2-4\mu}{2\rho_3}}\left(\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)\right)\right]^{\frac{1}{n+1}} \times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2-2a_2\mu+\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}, \tag{49}$$

when $\eta^2 - 4\mu < 0$, we have the following trigonometric function solution

$$\psi(x,t) = \left[\mp\sqrt{\frac{\eta^2-4\mu}{2\rho_3}}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)\right]^{\frac{1}{n+1}} \times \quad e^{i(Jx+(k_1+e_1-h_1J-a_1J^2-2a_1\mu+\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \tag{50}$$

$$\phi(x,t) = \left[\mp\sqrt{\frac{\eta^2-4\mu}{2\rho_3}}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)\right]^{\frac{1}{n+1}} \times \quad e^{i(Jx+(k_2+e_2-h_2J-a_2J^2-2a_2\mu+\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \tag{51}$$

For $c_1 \neq 0,\ c_2 = 0$ we have

$$\psi(x,t) = \left[\mp\sqrt{\frac{\eta^2-4\mu}{2\rho_3}}\left(\tan\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)\right)\right]^{\frac{1}{n+1}} \times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2-2a_1\mu+\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}, \tag{52}$$

$$\phi(x,t) = \left[\mp\sqrt{\frac{\eta^2-4\mu}{2\rho_3}}\left(\tan\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)\right)\right]^{\frac{1}{n+1}} \times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2-2a_2\mu+\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}, \tag{53}$$

when $\eta^2 - 4\mu = 0$, we have the following rational function solution

$$\psi(x,t) = \left[\mp\frac{1}{\sqrt{2\rho_3}}\left(\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^\alpha}\right)\right]^{\frac{1}{n+1}} \times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2-2a_1\mu+\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \tag{54}$$

$$\phi(x,t) = \left[\mp\frac{1}{\sqrt{2\rho_3}}\left(\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^\alpha}\right)\right]^{\frac{1}{n+1}} \times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2-2a_2\mu+\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \tag{55}$$

For $C_1 = 0,\ C_2 \neq 0$ we have

$$\psi(x,t) = \left[\mp\frac{1}{\sqrt{2\rho_3}}\left(\frac{\Gamma(1+\alpha)}{\zeta^\alpha}\right)\right]^{\frac{1}{n+1}} \times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2-2a_1\mu+\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \tag{56}$$

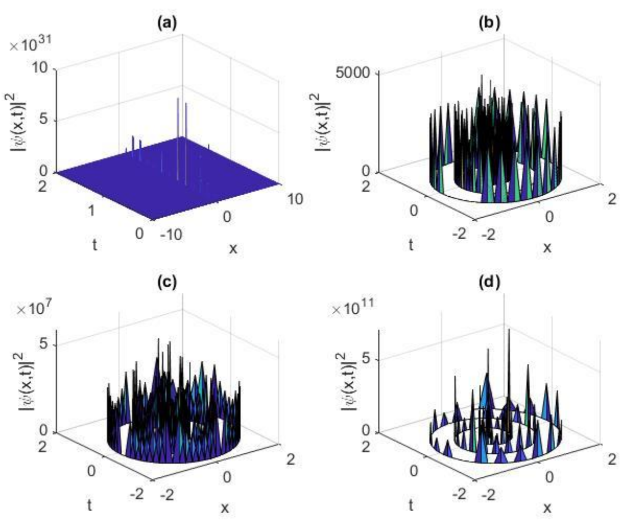

Figure 2: The solution |ψ(x,t)|^2 of Eq.(56) with a=0.5, J=0.5, n=0, α=1, b_3=1, η=3, μ=0.5, x∈[-10,10], t∈[0,2], and α=1 for Figure (a) and in cylindrical coordinates α=0.25 for Figure (b), α=0.5 for Figure (c) and α=0.75 for Figure (d).

$$\phi(x,t) = \left[\mp\frac{1}{\sqrt{2\rho_3}}\left(\frac{\Gamma(1+\alpha)}{\zeta^\alpha}\right)\right]^{\frac{1}{n+1}} \times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2-2a_2\mu+\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \tag{57}$$

Case 2: when $B_0 = 0$, $B_1 = \mp\frac{A_0\sqrt{-2\rho_3}}{2\mu}$, $A_0 = A_0$, $A_1 = \frac{A_0\eta}{2\mu}$, $\rho_1 = \frac{\eta^2}{2} - 2\mu$, $\rho_2 = 0$. Provide $\mu \neq 0$. As a result, Eq.(2) has the following traveling wave solutions:

when $\eta^2 - 4\mu > 0$, we have the following hyperbolic function solution

$$\psi(x,t)=\left[\mp\frac{2\mu+\frac{\eta\sqrt{\eta^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta^2}{2}}{\sqrt{\frac{-\rho_3(\eta^2-4\mu)}{2}}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}} \times \quad e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \quad (58)$$

$$\phi(x,t)=\left[\mp\frac{2\mu+\frac{\eta\sqrt{\eta^2-4\mu}}{2}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta^2}{2}}{\sqrt{\frac{-\rho_3(\eta^2-4\mu)}{2}}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}} \times \quad e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \quad (59)$$

For $c_1 \neq 0$, $c_2 = 0$ we have

$$\psi(x,t)=\left[\mp\frac{2\mu+\frac{\eta\sqrt{\eta^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta^2}{2}}{\sqrt{\frac{-\rho_3(\eta^2-4\mu)}{2}}\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}} \times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}, \quad (60)$$

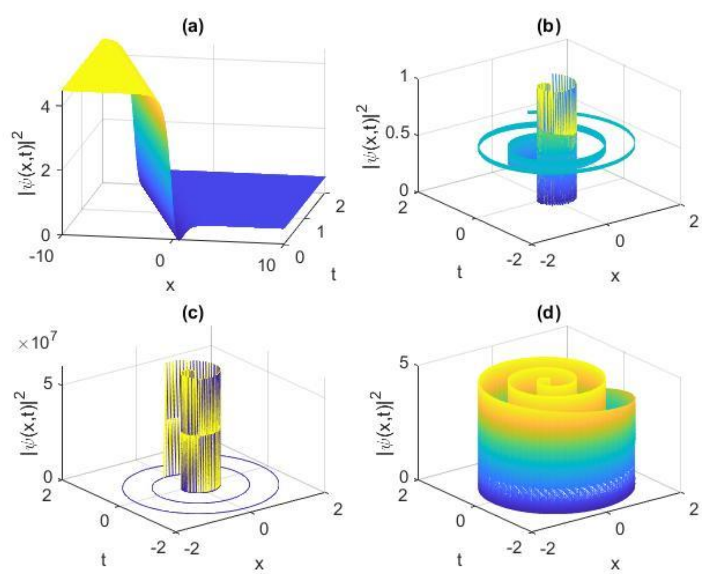

Figure 3: The solution |ψ(x,t)|^2 of Eq.(60) with a=-0.5, J=3.5, n=0, b_3=-1.01, η=4, μ=3, x∈[-10,10], t∈[0,2], and α=1 for Figure (a) and in cylindrical coordinates α=0.25 for Figure (b), α=0.5 for Figure (c) and α=0.75 for Figure (d).

$$\phi(x,t)=\left[\mp\frac{2\mu+\frac{\eta\sqrt{\eta^2-4\mu}}{2}\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta^2}{2}}{\sqrt{\frac{-\rho_3(\eta^2-4\mu)}{2}}\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}, \tag{61}$$

when $\eta^2-4\mu<0$, we have the following trigonometric function solution

$$\psi(x,t)=\left[\mp\frac{2\mu+\frac{\eta\sqrt{4\mu-\eta^2}}{2}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta^2}{2}}{\sqrt{\frac{-\rho_3(4\mu-\eta^2)}{2}}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times\ e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \tag{62}$$

$$\phi(x,t)=\left[\mp\frac{2\mu+\frac{\eta\sqrt{4\mu-\eta^2}}{2}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta^2}{2}}{\sqrt{\frac{-\rho_3(4\mu-\eta^2)}{2}}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times\ e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \tag{63}$$

For $c_1\neq 0,\ c_2=0$ we have

$$\psi(x,t)=\left[\mp\frac{2\mu+\frac{\eta\sqrt{4\mu-\eta^2}}{2}\tan\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta^2}{2}}{\sqrt{\frac{-\rho_3(4\mu-\eta^2)}{2}}\tan\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}, \tag{64}$$

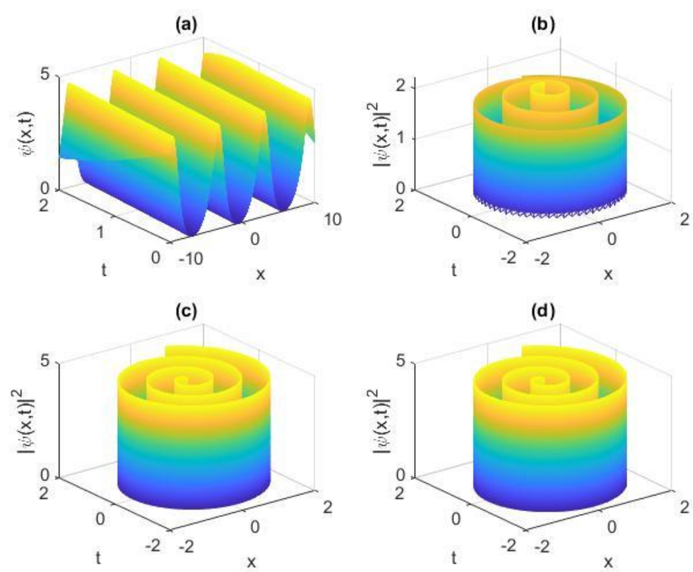

Figure 4: The solution |ψ(x,t)|^2 of Eq.(64) with a=0.5, J=0.5, n=0, b_3=1, η=3, μ=2.5, x∈[-10,10], t∈[0,2], and α=1 for Figure (a) and in cylindrical coordinates α=0.25 for Figure (b), α=0.5 for Figure (c) and α=0.75 for Figure (d).

$$\phi(x,t)=\left[\mp\frac{2\mu+\frac{\eta\sqrt{4\mu-\eta^2}}{2}\tan\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)-\frac{\eta^2}{2}}{\sqrt{\frac{-\rho_3(4\mu-\eta^2)}{2}}\tan\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}, \quad (65)$$

when $\eta^2-4\mu=0$, we have the following rational function solution

$$\psi(x,t)=\left[\mp\frac{2\mu+\eta\left(\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^{\alpha}}\right)-\frac{\eta^2}{2}}{\sqrt{-2\rho_3}\left(\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^{\alpha}}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \quad (66)$$

$$\phi(x,t)=\left[\mp\frac{2\mu+\eta\left(\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^{\alpha}}\right)-\frac{\eta^2}{2}}{\sqrt{-2\rho_3}\left(\frac{C_2\Gamma(1+\alpha)}{C_1\Gamma(1+\alpha)+C_2\zeta^{\alpha}}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \quad (67)$$

For $C_1=0,\ C_2\neq 0$ we have

$$\psi(x,t)=\left[\mp\frac{2\mu+\eta\left(\frac{\Gamma(1+\alpha)}{\zeta^{\alpha}}\right)-\frac{\eta^2}{2}}{\sqrt{-2\rho_3}\left(\frac{\Gamma(1+\alpha)}{\zeta^{\alpha}}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \quad (68)$$

$$\phi(x,t)=\left[\mp\frac{2\mu+\eta\left(\frac{\Gamma(1+\alpha)}{\zeta^{\alpha}}\right)-\frac{\eta^2}{2}}{\sqrt{-2\rho_3}\left(\frac{\Gamma(1+\alpha)}{\zeta^{\alpha}}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \quad (69)$$

Case 3: when $B_0=\mp\frac{A_0\eta\sqrt{-2\rho_3}}{\eta^2-4\mu},\ B_1=\mp\frac{2A_0\sqrt{-2\rho_3}}{\eta^2-4\mu},\ A_0=A_0,\ A_1=0,\ \rho_1=\frac{\eta^2}{2}-2\mu,\ \rho_2=0$. Provide $\eta^2-4\mu\neq 0$. As a result, Eq.(2) has the following traveling wave solutions:

when $\eta^2-4\mu>0$, we have the following hyperbolic function solution

$$\psi(x,t)=\left[\mp\frac{\eta^2-4\mu}{\sqrt{-\rho_3(\eta^2-4\mu)}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{2}\right)}\right)}\right]^{\frac{1}{n+1}}\times\ e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \quad (70)$$

$$\phi(x,t)=\left[\mp\frac{\eta^2-4\mu}{\sqrt{-\rho_3(\eta^2-4\mu)}\left(\frac{c_1\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)+c_2\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)}{c_1\cosh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)+c_2\sinh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{2}\right)}\right)}\right]^{\frac{1}{n+1}}\times\ e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \quad (71)$$

For $c_1\neq 0,\ c_2=0$ we have

$$\psi(x,t)=\left[\mp\frac{\eta^2-4\mu}{\sqrt{-\rho_3(\eta^2-4\mu)}\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^{\alpha}\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}, \quad (72)$$

$$\phi(x,t)=\left[\mp\frac{\eta^2-4\mu}{\sqrt{-\rho_3(\eta^2-4\mu)}\tanh\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)-\frac{\eta}{2}}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}, \quad (73)$$

when $\eta^2-4\mu<0$, we have the following trigonometric function solution

$$\psi(x,t)=\left[\mp\frac{\eta^2-4\mu}{\sqrt{-\rho_3(4\mu-\eta^2)}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}. \quad (74)$$

$$\phi(x,t)=\left[\mp\frac{\eta^2-4\mu}{\sqrt{-\rho_3(4\mu-\eta^2)}\left(\frac{c_1\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)+c_2\sin\left(\frac{\sqrt{4\mu-\eta^2}}{2\Gamma(1+\alpha)}\zeta^2\right)}\right)}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}. \quad (75)$$

For $c_1\neq 0,\ c_2=0$ we have

$$\psi(x,t)=\left[\mp\frac{\eta^2-4\mu}{\sqrt{-\rho_3(\eta^2-4\mu)}\tan\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_1+e_1-h_1J-a_1J^2+2a_1\mu-\frac{a_1\eta}{2})\frac{t}{\alpha}+k)}, \quad (76)$$

$$\phi(x,t)=\left[\mp\frac{\eta^2-4\mu}{\sqrt{-\rho_3(\eta^2-4\mu)}\tan\left(\frac{\sqrt{\eta^2-4\mu}}{2\Gamma(1+\alpha)}\zeta^\alpha\right)}\right]^{\frac{1}{n+1}}\times e^{i(Jx+(k_2+e_2-h_2J-a_2J^2+2a_2\mu-\frac{a_2\eta}{2})\frac{t}{\alpha}+k)}, \quad (77)$$

7. Conclusions

In this paper, the FNLSE with generalized AC nonlinearity was found to have optical soliton solutions in fiber BGs. The conformable fractional derivative definition is utilized. Finally, the solutions were retrieved using the rational Fractional ($\frac{D_\zeta^\alpha G}{G}$)-expansion Method.

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