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| RESEARCH ARTICLE

On the Picture-Perfect Number

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| ABSTRACT
This is a seminar paper on the article picture perfect numbers by Joseph L. Pe that was published in the journal mathematical spectrum in 2008. This paper begins with a discussion of the definition of the more familiar concept of perfect numbers, then proceeds to a discussion of the picture-perfect numbers as defined by Joseph L. Pe, and winds up with a discussion on how to obtain a picture-perfect number using the Andersen's Theorem. This paper also includes proof of the Andersen's Theorem, as well as that of the Andersen's Lemma, which are both attributed to Jens Kruce Andersen.

| KEYWORDS

Picture-Perfect Number, Project Paper, Mathematics, Andersen's Theorem, Andersen's Lemma

| ARTICLE INFORMATION

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1. Introduction

For centuries, the properties and patterns of numbers have held a deep fascination for both mathematicians and those passionate about mathematics. One such fascination is centered on those natural numbers that are equal to the sum of all of their proper divisors. The smallest example is 6 since $6 = 1 + 2 + 3$. The number 28 is another natural number that shares this same property: $28 = 1 + 2 + 4 + 7 + 14$. Indeed, these kinds of numbers are unique, for in the set of positive integers, there are only a few of them. Here are the first six natural numbers that share unique property: 6, 28, 496, 8128, 33550336, 8589869056. Such numbers are called "perfect" numbers.

The pursuit of *perfect numbers* dates back to ancient times (Shanks, 1978). The first three perfect numbers (6, 28, and 496) were known to the ancient mathematicians since Pythagoras (circa 500 BC). In the paper of Voight (1998). he said that these *perfect numbers* have seen much mathematical study. Indeed, many of the fundamental theorems of number theory stem from the investigation of the Greeks into the problem of *perfect numbers* and *Pythagorean triples*.

The fascinating idea of *picture-perfect numbers* arose from the concept of *perfect numbers*. The beauty of a picture-perfect number lies in the property that its reverse is equal to the sum of the reverses of its proper factors. *Picture numbers* are also known as *mirror-perfect numbers*. This alternative label springs from the fact that a mirror projects the exact linage of an object in front of it. Pe (2008) formally defines a picture-perfect number as a number *n* such that the reverse of *n* equals the sum of the reverses of the divisors of *n*.

In this definition, Pe uses the word *number* to refer to any natural number, and in this seminar, we will do the same.

This seminar paper exposes the definition and the discovery of picture-perfect numbers, Andersen's Lemma, Andersen's Theorem, and their proofs, as presented by Joseph L. Pe in his paper "The Picture-Perfect Numbers." The last sections of Pe's article on *conjectures and extensions, picture-perfect semi-primes,* and *picture amicable pairs* are beyond the scope of this seminar paper since such exposition of these latter topics would require further and thorough investigation.

2. Literature Review

2.1 The Perfect Numbers

In the article "Perfect Number: An Elementary Introduction", published in 1998. Voight clarified the definition of a perfect number by formally defining the sum of divisors function. He defines the sum of divisors function as $\sigma(n) = \sum_{d|n} d$, where *d* runs over the positive divisors of n, including 1 and *n* itself. For example,

$$\sigma\ (11) = 1 + 11 = 12 \text{ and } \sigma\ (15) = 1 + 3 + 5 + 15 = 24.$$

He continued by stating that the number *N* is perfect if $\sigma(N) = 2N$. He added $\sigma(N) < 2N$, then *N* is called a deficient number; otherwise, when $\sigma(N) > 2N$, he said it is called an abundant number. He further specified that a perfect number *N* is equivalent to saying that the sum of the proper (or aliquot/ portion) divisors of *N* is equal to *N*. For simplicity, a positive integer (natural number) is said to be a perfect number if it is equal to the sum of all of its positive factors, excluding itself.

Example

$$6 = 1 + 2 + 3 = 6$$
$$28 = 1 + 2 + 4 + 7 + 14 = 28$$

This suggests that the smallest perfect number is 6 while the largest is still unknown. Researchers are still working so hard to generate this number, leading to Mersenne prime.

2.2 The Elusive Picture-Perfect Numbers

In his exploration of perfect numbers, Pe incidentally noted something exceptional about the natural number 10311. In the set of perfect numbers {6, 28, 496, 8128, 33550336, 8589869056,...}, there is no way you will encounter number 10311. As recall, a perfect number is a number whose sum of proper divisors is equal to itself. Clearly. 10 311 does not satisfy this requirement. Note that the proper divisors of 10311 are 1, 3, 7, 21, 491, 1473, and 3437, which sum up to 5433 and not 10311. That is,

$$10311 \neq 1 + 3 + 7 + 21 + 491 + 1473 + 3437$$
$$= 5\,433$$

Since the sum 5 433 is less than $n = 10311$, by definition of deficient number, 10311 is said to be deficient; hence, 10311 is not a perfect number. Pe, however, was able to note there is something exceptional about the number 10311. Observe that if we get the reverse of 10311 as well as the reverse of each of its proper divisors and get the sum of reverses of these divisors, then we will have

$$7343 + 3741 + 194 + 12 + 7 + 3 + 1 = 11301$$

This result yields the idea of *picture-perfect* or *mirror-perfect numbers*. Pe called a number *n picture-perfect* or *mirror-perfect* if the reverse of *n* is *denoted by* $f(n)$. is equal to the sum of the reverse of the proper divisors of *n*, denoted by $R(n)$, thus $f(n) = R(n)$. In other words, $f(n)\ f(d_1) + f(d_2) + ...$ where the $d_i's$ are tix proper divisors of n. Note that if the reverse of a particular divisor of *n* is headed by zero/zeroes, we disregard the zero/zeroes. This is so because if we reverse, for example, integer 120, we get 021, where 0 has no significant value; disregard it. If we place *n* on the left side of an equation, the unevaluated sum of the proper divisors of *n* is placed on the other side, then the resulting equation, if read backward, becomes valid. Take, for example, the $10311 = 1 + 3 + 7 + 21 + 491 + 1473 + 3437$ is invalid. However, when read backward, the equation is valid,

$$7343 + 3741 + 194 + 12 + 7 + 3 + 1 = 11301$$

This explains the use of the term "picture-perfect". since a picture of an object is a mirror image (i.e., an orientation reversal) of that object. Please take note, however, that since addition is commutative, changing the order of the reverses of the proper divisors of n will not affect their sum. The above manner of ordering the proper divisor is employed to be consistent with mirror effects as we subscribe to the idea that a *picture-perfect number* is also known as a *mirror-perfect number*.

The first *picture-perfect number* (also denoted as ppn) is not 10311. It is 6. Recall that 6 is also the first perfect number since $6 = 1 + 2 + 3$, where l, 2, and 3 are its proper divisors. To show that 6 is a ppn (picture-perfect number), the reverse of 6 must be equal to the sum of the reverses of its proper divisors, $f(6) = R(6)$, so we have the equation $3 + 2 + 1 = 6$, again we reverse the sequence of the divisors just to be consistent with the mirror effects. It shows a valid equation. 6 is a picture-perfect number.

As you notice, 6 has only single-digit proper divisors, making 6 a trivial ppn. The search of such kind of numbers, after 6, yielded the next ppn 10311. Thus the first non-trivial ppn is 10311.

Pe introduced the equation $P =_b D$ as the *mirror equation* of P if *P* is a ppn, where *D* is the (unevaluated) sum of the proper divisors of *P*. The symbol $=_b$ indicates that the mirror equation should be read backward to be valid. Thus, the mirror equation for $P = 10311$ is written in the form

$$10311 =_b \ 1 + 3 + 7 + 21 + 491 + 1473 + 3437$$

Pe mentioned that there is a specialized code in Mathematica, a mathematical software, to generate picture-perfect numbers not exceeding 10^{10}:

```
f[n_] := FromDigits[Reverse[IntegerDigits[n]]];
n = 2; (*Initial value of n*)
While[n<10^10, If[f[n]==Apply[Plus,Map[f, Drop[Divisors[n], -1]]], Print[n]];n++]
```

Note, however, that this specialized code in generating ppns is beyond the scope of this paper.

Pe almost conjectured that 10311 was the only non-trivial ppn after his computer search showed no new ppn less than 10^7. However, after several hours of *Mathematica* running on his machine, Pe was rewarded with the third ppn, 21661371, which has a mirror equation

$$21661371 =_b \ 1 + 3 + 9 + 27 + 443 + 1329 + 1811 + 3987 + 5433 + 11961 + 16299 + 48897 + 802273 + 2406819 + 7220457$$

The discovery of Pe of this large ppn resulted in high hopes that the fourth ppn number would be found only briefly. This prompted the two problem enthusiasts, Daniel Dockery and Mark Ganson, to join him in the search. They came up with their discussion forum, which they use as their medium to update each other on new ideas and results they may have, including their software customized for the search of ppn. Pe mentioned that Ganson provided a Windows search utility.

Pe, Dockey, and Ganson focused their search of ppn on the interval from 10^9 to 10^{10}. Three weeks later, after Pe found the third ppn, Dockery found the fourth ppn, which is already in the ten-digit $P = 1\,460\,501\,511$ with the mirror equation of

$$1460501511 =_b 1 + 3 + 7 + 21 + 101 + 303 + 707 + 2121 + 688591 + 2065773 + 4820137 + 14460411 + 69547691 + 208643073 + 486833837.$$

After Dockery found the fourth ppn their search became slow, even when using Ganson's new search program C++, which runs twice as fast as the *Mathematica*.

Pe named the newest member of their forum, Jens Kruse Andersen, who was able to generate the next ppn with $P = 79\,800\,620$ with mirror equation,

$$7980062073 =_b 1 + 3 + 19 + 57 + 140001089 + 420003267 + 2660020691$$

Pe mentioned that Andersen was able to formulate a more efficient algorithm that caches divisor information. Pe, however, did not elaborate nor present the exact form of said algorithm. Nevertheless, indeed, this must have been a helpful algorithm. In fact, after arriving at this new and efficient algorithm, Andersen tested all numbers up to 10^{10} and concluded that there are no more ppn's within this range other than those five that were discovered already.

For over a month, Pe, together with the others who tried to search for more ppn, used Andersen's algorithm and exhausted the search for ppn in the interval from 10^{10} to 10^{12} to add to the first five ppn they generated from 10^{10} to 10^{12} the sixth and the seventh ppn's: The search 6, 10311, 21661371, 1460501511, 7980062073, **79862699373**, **798006269373**, … . The search for more ppn's continues today.

Pe idiomatically expresses his thought on the search for ppn, "Small ppn's are rare pearls in the infinite ocean of numbers." There are only five ppn's below 10^{10}. The seven ppn's listed above are the only ppn's less than 10^{12}. At the time of Pe's writing of his article on ppn, the value of the eighth ppn was still unknown.

2.3 Andersen's Theorem

Euclid said all perfect numbers should take the expression $2^{n-1}(2^n - 1)$. Leonard Euler proved in his posthumous paper that if N is an even perfect number, then N can be written in the form $N = 2^{n-1}(2^n - 1)$, where $(2^n - 1)$ is prime. Now, since the idea of a picture-perfect number originated from the concept of the perfect number, Andersen, one of the four ppn enthusiasts mentioned above, discovered a remarkable result, as stated by Pe in his article.

Andersen's Theorem: If the natural number $p = 140z10n89$ is prime, then the product 57 p is picture-perfect, and conversely, where *z* is any string (possibly none) of 0's and n is any string (possibly none) of 9's.

For example, in 140**00**1089, *z* is the **00** after 140. Also, for the number 14010**999**89, *n* is the **999** immediately before 89. This was conceptualized by Jens Andersen (the proponent of this theorem) as a product of his exploration relative to his search for ppn using a supercomputer. He said that if *n* in *p* has no 9s, 57 *p* has the form $798z62073$. For example, 57 *p* = 798**00**62073 when *p* = 140**00**1089 since *n* has no 9s in p. Likewise, if *n* has at least one 9, then 57 p has the form 798*z*626*m*373. For example, 57 p = 798626**99**373 when *p* = 14010**999**89, whose *n* consists of three 9s and *m* in 57 *p* has one 9 less than *n*. This means the product of 57 *p*, where the *n* of *p* is at least one, consists of at least one 9 but is always less than the number of 9s in *p*. The proof of Andersen's theorem appears towards the end of this section, as we will encounter Andersen's lemma. If you notice, the product of 57 p is with space between the multiplicands, and it is because Andersen, according to Pe, emphasizes the linking together of digits such as 140*z*10*n*89.

Let us call a prime *p* of the form 140*z*10*n*89 an *Andersen prime*, and its corresponding picture- perfect number 57 *p* an *Andersen number*. In the table below, we show that for every Andersen prime p, there corresponds to a *picture-perfect number* obtained by multiplying 57 by the Andersen prime. This particular kind of *picture-perfect number* is referred to as an *Andersen number*. Examples of Andersen prime/ number pairs are:

Andersen Prime (p)	***Corresponding Andersen Number (57 p)***
140**00**1089	798**00**62073
14010**999**89	798626**99**373
140**00**10**99**89	798**00**626**9**373
140**000**10**99999**89	798**000**626**9999**373

The finite sequences *z*, *n*, and *m* (as defined above) appear in **boldface**.

After Andersen announces the result of his work, Ganson right away uses Andersen's result to find the 204 (primarily huge) Andersen numbers, the largest of which has 177 digits. He called it the gargantuan 798*z*626*m*373, where indeed only super computers can generate it because this Andersen number has 77 zeros (which is the *z*) and 91 nines (which is the m). With the collaborative work of Andersen and Ganson, thousands of Andersen numbers were discovered. At the time of Pe's writing of the article on ppn, the largest Andersen prime that they discovered had 2,461 digits, is $140 \times 10^{2458} + 1089$ and was verified as prime using *Primo* by Marcel Martin, a software verifier of discovered primes. This Andersen prime constitutes an Andersen Number Of the form $798 \times 10^{2459} + 62073$ equivalent digits of 2,461. Later, after the discovery of the largest Andersen prime, the two were able to find probable Andersen primes with more than 10^4 digits. However, these have not been tested yet because, currently, there is no way to test the primality of numbers of this size.

The proof of Andersen's theorem is credited to Andersen himself. This proof uses **Andersen's Lemma**, which states that "If *p* (not necessarily prime) is of the form 140*z*10*n*89, then $R(57p) = 170 + R(p) + R(3\,p) + R(19\,p)$." Note that Joseph Pe proved the Andersen Lemma. Before we prove the Andersen Theorem, we first establish the Andersen Lemma.

Andersen's Lemma. If *p* (not necessarily prime) is of the form 140*z*10*n*89, then $R(57\,p) = 170 + R(p) + p) + R(19\,p)$.

In establishing the Andersen's Lemma, Pe considered the following two cases:
Case (1)

Suppose *n* has at least one 9, then *p*= 140*z*10*n*89. Thus, *p* follows the finite sequence of *z*, *n*, and *m* as defined previously (e.g., 57 *p* = 798626**99**373 whose *p* = 14010**999**89 whose *n* consists of three 9s), which appeared in **boldface** on the table above. If we simplify the notation by clustering the subscript of *z*, *n,* and *m*, as in the table above, we have 00, 999, and 99 as *z*, *n,* and *m*, respectively. There is no loss of generality in this claim, as the boldface finite sequences can be replaced by an arbitrary sequence in the appropriate manner. (e.g., we will make the zeros into three; the same goes for n, which will have four 9s, and also, the product m will become three 9s). In other words, the increasing of the number of *z*, *n*, and *m* does not affect the correctness of the proof as long as they increase respectively; that is, if you increase the number of zeros, the same goes for n and m.

To be convinced of this generalization, below is the manual computation of the product of 57 *p*

$$\begin{array}{r} 140001099989 \\ \times \qquad\qquad 57 \\ \hline 980007699923 \\ 700005499945 \\ \hline 7980062699373 \end{array} \qquad \begin{array}{r} 140001099989 \\ \times \qquad\qquad 19 \\ \hline 1260009899901 \\ 140001099989 \\ \hline 2660019899791 \end{array} \qquad \begin{array}{r} 140001099989 \\ \times \qquad\qquad 3 \\ \hline 140003299967 \end{array}$$

Thus, with $p = 140001099989$, the products obtained are the following

$57p = 798\mathbf{00}626\mathbf{99}373$
$19p = 266\mathbf{00}208\mathbf{99}791$
$3p = 420\mathbf{00}32\mathbf{99}967$
$p = 140\mathbf{00}10\mathbf{999}89$

As we take the reverses of each product that is R(*n*), we have

$R(p) = 98\mathbf{999}01\mathbf{00}041$
$R(3\,p) = 76\mathbf{999}23\mathbf{00}024$
$R(19p) = 197\mathbf{99}802\mathbf{00}662$
$R(57\,p) = 373\mathbf{99}626\mathbf{00}897$

Adding *R*(19 *p*), *R*(3 *p*), *R*(*p*), and 170 gives *R*(57 *p*), as required to generate a perfect number, we have

$$\begin{array}{r} 197\mathbf{99}802\mathbf{00}662 \\ 76\mathbf{999}23\mathbf{00}024 \\ + \quad 98\mathbf{999}01\mathbf{00}041 \\ 170 \\ \hline 373\mathbf{99}626\mathbf{00}897 \end{array}$$

As you notice, the sum of $R(19\,p) + R(3p) + R(p) + 170 = R(57\,p)$.

Case (2)

If *n* has no 9s, that is *p* = 140*z*1089. We do precisely what we did in case 1. It is just that towards the end, the summation $of\ R(19\,p) + R(3\,p) + R(p) + 170 = R(57\,p)$, using the same notation as case (1) we have,

$$\begin{array}{r} 19602\mathbf{00}662 \\ 7623\mathbf{00}024 \\ + \quad 9801\mathbf{00}041 \\ 170 \\ \hline 37026\mathbf{00}897 \end{array}$$

These results display the beauty of this number. Truly, its property is indeed unique.
Finally, we prove the Andersen Theorem using the Andersen Lemma.
(⇒) Suppose that *p* is prime. Then the proper divisors of 57 $p = 3 \times 19 \times p$ are

1, 3, 19, 57, *p*, 3 *p*, 19 *p*. (D)

Getting the R(n), that is, the sum of the reverses of the proper divisors of 57 p, we have

$R(1) + R(3) + R(19) + R(57) + R(p) + R(3p) + R(19\,p)$
$= (1 + 3 + 91 + 75) + R(p) + R(3p) + R(19p)$
$= 170 + R(p) + R(3p) + R(19p)$
$= R(57p)$

Therefore, by the Andersen Lemma, 57 *p* is picture-perfect.
(⇐) Suppose 57 p is picture-perfect but p is not prime, then 57 *p* will have more divisors than 1, 3, 19, 57, *p*, 3 *p*, 19 *p*, then consequently the sum of the reversed divisors becomes larger than $170 + R(p) + R(3\,p) + R(19\,p) = R(57\,p)$. Hence, 57 *p* cannot be a picture-perfect number. This establishes the Andersen Theorem.

3. Conclusions

As quoted from Joseph Pe's article,

> "The sequence of picture-perfect numbers is an example of a sequence that appears at first to be extremely sparse, even finite, but yields many terms in the scale of the very large. Indeed, the sequence of ppn's has been compared to what first

appears to be a faint star in the universe of numbers, but is then revealed to be an abundant galaxy by the computer-telescope."

The *picture-perfect number* is one of the fascinating discoveries relative to the unique properties of natural numbers. As the set of natural numbers is infinite, the search and discovery of more picture-perfect numbers will continue. We need more enthusiasts like Andersen who will be interested in finding more of these rare kind of numbers. Indeed, the search for *picture-perfect numbers* is complex, for there is no such explicit indicator (at least not at the moment) that characterizes a picture-perfect number. Presently, what the ppn enthusiasts do is run the search through the use of supercomputers to generate numbers of this kind based on the definition. The search is difficult, for there are only a few of them on the smaller scale, but more on the larger scale Of the set of natural numbers — so much so that the search may be likened to toting to unearth black pearls beneath the deep blue sea. It appears few on the surface, but if you want to find more, you must go deeper.

Since picture-perfect numbers are as elusive as gigantic prime numbers, they are a suitable alternative to using prime numbers in cryptography, where large prime numbers are employed to encrypt and decrypt highly classified messages.

The following exciting conjectures and open problems from the article are as a result of this recommended for further explorations of picture-perfect numbers:

i. (Conjecture - Ganzon) All *picture-perfect numbers* are divisible by 3 (Ganzon);
ii. (Conjecture - Pe & Ganzon) Every non-*trivial picture-perfect number* is odd;
iii. (Open Problem – Pe) Whether the first non-trivial *picture-perfect number* (10311) may be used to generate other *picture-perfect numbers* in the same way that the first Andersen number (7980062073) does.

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