Journal of Mathematics and Statistics Studies
ISSN: 2709-4200
DOI: 10.32996/jmss
Journal Homepage: www.al-kindipublisher.com/index.php/jmss

JMSS
AL-KINDI CENTER FOR RESEARCH AND DEVELOPMENT

| RESEARCH ARTICLE

The Carnot Theorem in Einstein Gyrovector Spaces

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| ABSTRACT
In Euclidean geometry, Carnot's theorem is a direct application of the theorem Pythagoras. In [4,6] A.A. Ungar, employs the Einstein gyrovector spaces for the introduction of the gyrotrigonometry. Ungar's and other researcher's works play a major role in translating some theorems from Euclidean geometry to corresponding theorems in Einstein gyrovector spaces.In [2] Demirel and Soytürk proved that hyperbolic Carnot theorem. In this paper, we present Carnot's theorem in Einstein's gyrovector spaces in terms of gamma factors.

| KEYWORDS

Gyrotriange, Einsteinian-Phythagorean Identities, Gamma Factor, Carnot Theorem

| ARTICLE INFORMATION

ACCEPTED: 05 February 2024 **PUBLISHED:** 27 February 2024 **DOI:** 10.32996/jmss.2024.5.1.3

1. Introduction

Hyperbolic geometry appeared in the first half of the 19th century. It is also known as a type of non-Euclidean geometry. Although Euclidean Geometry and Hyperbolic Geometry have common concepts such as distanceand angle, both these geometries have many differences. Hyperbolic Geometry has many models, such as: Poincare' disc model, Einstein's relativistic velocity model, etc.

Einstein gyrovector spaces form the algebraic setting for the Beltrami-Klein ball model of Hyperbolic Geometry, just as vector spaces form the algebraic setting for the standard model of Euclidean Geometry.

Let c be the vacuum speed of light, and let

$$\mathbb{R}_c^3 = \{\boldsymbol{v} \in \mathbb{R}^3 : \|\boldsymbol{v}\| < c\} \tag{1.1}$$

be the c ball of all relativistically admissible velocities of material particles. Einstein's addition in c-ball is given by the equation.

$$\boldsymbol{u} \oplus \boldsymbol{v} = \frac{1}{1+\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}}\left\{\boldsymbol{u} + \boldsymbol{v} + \frac{1}{c^2}\frac{\gamma_{\boldsymbol{u}}}{1+\gamma_{\boldsymbol{u}}}\big(\boldsymbol{u} \times (\boldsymbol{u} \times \boldsymbol{v})\big)\right\} \tag{1.2}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_c^3$, where $\boldsymbol{u}\cdot\boldsymbol{v}$ is the inner product that the ball $\mathbb{R}_c^3$ inherits from its space $\mathbb{R}^3$, $\boldsymbol{u} \times \boldsymbol{v}$is the vector product in $\mathbb{R}_c^3 \subset \mathbb{R}^3$ and where $\gamma_{\boldsymbol{u}}$ is the gamma factor

$$\gamma_{\boldsymbol{u}} = \frac{1}{\sqrt{1-\frac{\|\boldsymbol{u}\|^2}{c^2}}} \geq 1 \tag{1.3}$$

in the *c*-ball.

Owing to the vector identity,

$$(\boldsymbol{x}\times\boldsymbol{y})\times\boldsymbol{z} = -(\boldsymbol{y}\cdot\boldsymbol{z})\boldsymbol{x} + (\boldsymbol{x}\cdot\boldsymbol{z})\boldsymbol{y} \tag{1.4}$$

for all $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\in\mathbb{R}^3$, Einstein addition (1.2) can also be written in the form

$$\boldsymbol{u}\oplus\boldsymbol{v} = \frac{1}{1+\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}}\{\boldsymbol{u}+\frac{1}{\gamma_{\boldsymbol{u}}}\boldsymbol{v}+\frac{1}{s^2}\frac{\gamma_{\boldsymbol{u}}}{1+\gamma_{\boldsymbol{u}}}(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{v}\} \tag{1.5}$$

which remains valid in higher dimensions. Einstein's addition (1.5) of relativistically admissible velocities was introduced by Einstein in 1905.

In this paper, we study an Einstein gyrovector space thatwas introduced by A. A. Ungar[see 4,5,6].

2. Preliminaries

Definition 2.1. A groupoid $(\mathbb{G},\oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In $\mathbb{G}$, there is at least one element, **0,** called left identity, satisfying.

$$\mathbf{0}\oplus\boldsymbol{a} = \boldsymbol{a}$$

for all $\boldsymbol{a}\in\mathbb{G}$. There is an element $\mathbf{0}\in\mathbb{G}$ for each $\boldsymbol{a}\in\mathbb{G}$ there is an element $\ominus\boldsymbol{a}\in\mathbb{G}$, called a left inverse of $\boldsymbol{a}$**,** satisfying

$$\ominus\boldsymbol{a}\oplus\boldsymbol{a} = \mathbf{0}.$$

Moreover, for any $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\in\mathbb{G}$, there exit a unique element $gyr[\boldsymbol{a},\boldsymbol{b}]\boldsymbol{c}\in\mathbb{G}$ such that binary operation obeys the left gyroassociative law

$$\boldsymbol{a}\oplus(\boldsymbol{b}\oplus\boldsymbol{c}) = (\boldsymbol{a}\oplus\boldsymbol{b})\oplus gyr[\boldsymbol{a},\boldsymbol{b}]\boldsymbol{c}.$$

The map $gyr\colon\mathbb{G}\to\mathbb{G}$ is given by $\boldsymbol{c}\mapsto gyr[\boldsymbol{a},\boldsymbol{b}]\boldsymbol{c}$ is an automorphism of the groupoid $(\mathbb{G},\oplus)$, that is,

$$gyr[\boldsymbol{a},\boldsymbol{b}]\in \boldsymbol{Aut}(\mathbb{G},\oplus)$$

and the automorphism $gyr[\boldsymbol{a},\boldsymbol{b}]$ of automorphism of $\mathbb{G}$ is called the gyroautomorphism of$\mathbb{G}$ generated by $\boldsymbol{a},\boldsymbol{b}\in\mathbb{G}$. Finally, the gyroautomorphism of $\mathbb{G}$ generated by $\boldsymbol{a},\boldsymbol{b}\in\mathbb{G}$ possesses the left loop property

$$gyr[\boldsymbol{a},\boldsymbol{b}] = gyr[\boldsymbol{a}\oplus b,\boldsymbol{b}].$$

Additionally, if the binary operation "$\oplus$ " obeys the gyrocommutative law

$$\boldsymbol{a}\oplus\boldsymbol{b} = gyr[\boldsymbol{a},\boldsymbol{b}](\boldsymbol{b}\oplus\boldsymbol{a})$$

for all $\boldsymbol{a},\boldsymbol{b}\in\mathbb{G}$, then $(\mathbb{G},\oplus)$ is called a gyrocommutative gyrogroup.

Definition 2.2. Let $\mathbb{V}$ be a real inner product space and let $\mathbb{V}_s$ be the s-ball of $\mathbb{V}$,

$$\mathbb{V}_s = \{\boldsymbol{v}\in\mathbb{V} : \|\boldsymbol{v}\| < s\},$$

where $s>0$ is an arbitrary fixed constant. Einstein addition $\oplus$ is a binary operation in $\mathbb{V}_s$ given by the equation

$$\boldsymbol{u}\oplus\boldsymbol{v} = \frac{1}{1+\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}}\{\boldsymbol{u}+\frac{1}{\gamma_{\boldsymbol{u}}}\boldsymbol{v}+\frac{1}{s^2}\frac{\gamma_{\boldsymbol{u}}}{1+\gamma_{\boldsymbol{u}}}(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{v}\}$$

where $\gamma_{\boldsymbol{u}}$ is the gamma factor

$$\gamma_{\boldsymbol{u}} = \frac{1}{\sqrt{1-\frac{\|\boldsymbol{u}\|^2}{s^2}}} \geq 1$$

in the s-ball $\mathbb{V}_s$, and where $\cdot$ and $\|.\|$ are the inner product and norm that the ball $\mathbb{V}_s$ inherits from its space $\mathbb{V}$.

Einstein addition satisfies the mutually equivalent gamma identities

$$\gamma_{\boldsymbol{u}\oplus\boldsymbol{v}} = \gamma_{\boldsymbol{u}}\gamma_{\boldsymbol{v}}\left(1+\frac{\boldsymbol{u}\cdot\boldsymbol{v}}{s^2}\right)$$

and

$$\gamma_{\ominus \boldsymbol{u} \oplus \boldsymbol{v}} = \gamma_{\boldsymbol{u}} \gamma_{\boldsymbol{v}} \left(1 - \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{s^2}\right)$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_s^n$.

When the nonzero vectors $\boldsymbol{u}$and $\boldsymbol{v}$in the ball$\mathbb{R}_s^n$of $\mathbb{R}^n$are parallel in $\mathbb{R}^n$, $\boldsymbol{u} \parallel \boldsymbol{v}$, that is, $\boldsymbol{u} = \lambda \boldsymbol{v}$for some $0 \neq \lambda \in \mathbb{R}$, Einstein addition reduces to the Einstein addition of parallel velocities

$$\boldsymbol{u} \oplus \boldsymbol{v} = \frac{\boldsymbol{u} + \boldsymbol{v}}{1 + \frac{\|\boldsymbol{u}\| \|\boldsymbol{v}\|}{s^2}}$$

Hence,

$$\|\boldsymbol{u}\| \oplus \|\boldsymbol{v}\| = \frac{\|\boldsymbol{u}\| + \|\boldsymbol{v}\|}{1 + \frac{\|\boldsymbol{u}\| \|\boldsymbol{v}\|}{s^2}}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}_s^n$. In this case, Einstein's addition is both commutative and associative.

In the Newtonian limit, $s \to \infty$, s-ball $\mathbb{R}_s^n$ expands to the whole of its space $\mathbb{R}^n$, and Einstein's addition $\oplus$ in $\mathbb{R}_s^n$ reduces to vector addition $+$ in $\mathbb{R}^n$.

Theorem2.3. ($\mathbb{R}_s^n$,$\oplus$) Einstein groupoid is a gyrocommutative gyrogroup.

Some gyrocommutative gyrogroups admit scalar multiplication, giving rise to gyrovector spaces.

Definition 2.4. A ($\mathbb{G}$,$\oplus$,$\otimes$) gyrovector space is a gyrocommutative gyrogroup ($\mathbb{G}$,$\oplus$) that obeys the following axioms:

1. $gyr[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a} \cdot gyr[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{b}$for all points $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{G}$.
2. $\mathbb{G}$ admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points and $\boldsymbol{a} \in \mathbb{G}$:
   - $1 \otimes \boldsymbol{a} = \boldsymbol{a}$
   - $(r_1 + r_2) \otimes \boldsymbol{a} = (r_1 \otimes \boldsymbol{a}) \oplus (r_2 \otimes \boldsymbol{a})$
   - $(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a})$
   - $\frac{|r| \otimes \boldsymbol{a}}{\|r \otimes \boldsymbol{a}\|} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|} \boldsymbol{a} \neq \boldsymbol{0}$, $r \neq 0$
   - $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{a})$
   - $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = I$
3. Real vector space structure ($\|\mathbb{G}\|$,$\oplus$,$\otimes$) for the set $\|\mathbb{G}\|$ of one-dimensional "vectors"
   $$\|\mathbb{G}\| \coloneqq \{\mp \|\boldsymbol{a}\| : \boldsymbol{a} \in \mathbb{G}\} \subset \mathbb{R}$$
   with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{G}$,
   - $\|r \otimes \boldsymbol{a}\| = |r| \otimes \|\boldsymbol{a}\|$
   - $\|\boldsymbol{a} \oplus \boldsymbol{b}\| \leq \|\boldsymbol{a}\| \oplus \|\boldsymbol{b}\|$

Theorem 2.5. An Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is an Einstein gyrocommutative gyrogroup $(\mathbb{R}_s^n, \oplus)$ with scalar multiplication $\otimes$ given by

$$r \otimes \boldsymbol{v} = s \frac{\left(1 + \frac{\|\boldsymbol{v}\|}{s}\right)^r - \left(1 - \frac{\|\boldsymbol{v}\|}{s}\right)^r}{\left(1 + \frac{\|\boldsymbol{v}\|}{s}\right)^r + \left(1 - \frac{\|\boldsymbol{v}\|}{s}\right)^r} \frac{v}{\|\boldsymbol{v}\|} = stanh(rtanh^{-1} \frac{\|\boldsymbol{v}\|}{s}) \frac{v}{\|\boldsymbol{v}\|}$$

Definition 2.6. Let $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ be an Einstein gyrovector space. Its gyrometric is given by the gyrodistance function $d_{\oplus}: \mathbb{R}_s^n \times \mathbb{R}_s^n \longrightarrow \mathbb{R}^{\geq 0} \coloneqq \{r \in \mathbb{R}: r \geq 0\}$,

$$d_{\oplus}(\boldsymbol{a}, \boldsymbol{b}) = \|\ominus \boldsymbol{a} \oplus \boldsymbol{b}\| = \|\boldsymbol{b} \ominus \boldsymbol{a}\|$$

where $d_{\oplus}(\boldsymbol{a}, \boldsymbol{b})$ is the gyrodistance of $\boldsymbol{a}$ and $\boldsymbol{b}$**.**

The unique Einstein gyroline L_{AB} that passes two given points A and B in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is represented by the equation

$$L_{AB} = A \oplus (\ominus A \oplus B) \otimes t$$

$t \in \mathbb{R}$. Gyrolines in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ turn out to be well-known geodesic of the Beltrami Klein ball model of hyperbolic geometry.

3. Some Gyrotrigonometric Identities

Let $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$ be three distinct points and $\ominus \boldsymbol{A} \oplus \boldsymbol{B}$, $\ominus \boldsymbol{A} \oplus \boldsymbol{C}$be two rooted gyrovectors with a common tail $\boldsymbol{A}$. They include the gyroangle$\boldsymbol{\alpha} = \angle \boldsymbol{BAC} = \angle \boldsymbol{CAB}$, the radian measure of which is given by the equation

$$\boldsymbol{cos\alpha} = \frac{\ominus \boldsymbol{A} \oplus \boldsymbol{B}}{\|\ominus \boldsymbol{A} \oplus \boldsymbol{B}\|} \cdot \frac{\ominus \boldsymbol{A} \oplus \boldsymbol{C}}{\|\ominus \boldsymbol{A} \oplus \boldsymbol{C}\|}. \tag{3.1}$$

Definition 3.1. A gyrotriangle $\boldsymbol{ABC}$ in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is a object formed by the three points $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$, called the vertices of the triangle, and the gyrovectors $\ominus \boldsymbol{A} \oplus \boldsymbol{B}$, $\ominus \boldsymbol{B} \oplus \boldsymbol{C}$ and $\ominus \boldsymbol{C} \oplus \boldsymbol{A}$**,** called the sides of the triangle. These are respectively, the sides opposite to the vertices **C, A**and **B.** The gyrotriangle sides generate the three gyrotriangle gyroangles $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and$\boldsymbol{\gamma}$ at the respective vertices **A, B** and **C**.

Gyrotriangle gyroangle sum in hyperbolicgeometry is less than π. The difference, δ,

$$\delta = \pi - (\alpha + \beta + \gamma) \tag{3.2}$$

being the gyrotriangular defect.

Theorem 3.2. Let $\boldsymbol{ABC}$ be a gyrotriangle in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, with vertices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$, and sides $\boldsymbol{c} = \ominus \boldsymbol{A} \oplus \boldsymbol{B}, \boldsymbol{a} = \ominus \boldsymbol{B} \oplus \boldsymbol{C}$ and $\boldsymbol{b} =\ominus \boldsymbol{C} \oplus \boldsymbol{A}$**,** with gyroangles $\boldsymbol{\alpha}, \boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ at the vertices **A, B** and **C**. Then we have the law of cosines

$$\gamma_c = \gamma_a \gamma_b (1 - b_s c_s cos\gamma) \tag{3.3}$$

where $a = \|\boldsymbol{a}\|, b = \|\boldsymbol{b}\|, c = \|\boldsymbol{c}\|andb_s = b/s$, etc.

Definition 3.3. A right gyroangle γ is a gyroangle measuring $\frac{\pi}{2}$ radians.

Theorem 3.4. A gyrotriangle $\boldsymbol{ABC}$ in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ is a right gyrotriangle with gyrolegs $\boldsymbol{a}, \boldsymbol{b}$ and gyrohypotenuse $\boldsymbol{c}$, if and only if

$$\gamma_c = \gamma_a \gamma_b. \tag{3.4}$$

Theorem 3.5. Let $\boldsymbol{ABC}$ be a right gyrotriangle in an Einstein gyrovector space$\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ with the right gyroangle $\gamma = \pi/2$. Then we have two distinct Einsteinian-Phytagorean identities

$$a^2 + \left(\frac{\gamma_b}{\gamma_c}\right)^2 b^2 = c^2 \tag{3.5}$$

$$\left(\frac{\gamma_a}{\gamma_c}\right)^2 a^2 + b^2 = c^2 \tag{3.6}$$

with hypotenuse c and legs a and b.

4. Main Result

As an application of Einteinian-Pythagorian identities in Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, we verify the following theorem:

Theorem 4.1. Let $\boldsymbol{ABC}$ be a gyrotriangle in an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$, with vertices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$,and sides$\ominus \boldsymbol{A} \oplus \boldsymbol{B}$, $\ominus \boldsymbol{B} \oplus \boldsymbol{C}and\ominus \boldsymbol{C} \oplus \boldsymbol{A}$, and the points $\boldsymbol{S}, \boldsymbol{T}$and $\boldsymbol{R}$ be located on the sides$\ominus \boldsymbol{A} \oplus \boldsymbol{B}$, $\ominus \boldsymbol{B} \oplus \boldsymbol{C}$and $\ominus \boldsymbol{C} \oplus \boldsymbol{A}$ of the gyrotriangle $\boldsymbol{ABC}$ respectively.If the perpendiculars to the sides of the triangle at the points $\boldsymbol{S}, \boldsymbol{T}$ and $\boldsymbol{R}$ corcurrent, then

$$(\gamma_b \gamma_d)^2 \left[\gamma_a^2 a^2 \ominus \gamma_f^2 f^2\right] \oplus (\gamma_a \gamma_d)^2 \left[\gamma_c^2 c^2 \ominus \gamma_b^2 b^2\right] \oplus (\gamma_a \gamma_c)^2 \left[\gamma_e^2 e^2 \ominus \gamma_d^2 d^2\right] = 0$$

where $a = \|\ominus \boldsymbol{A} \oplus \boldsymbol{S}\|, b = \|\ominus \boldsymbol{S} \oplus \boldsymbol{B}\|, c = \|\ominus \boldsymbol{B} \oplus \boldsymbol{T}\|, d = \|\ominus \boldsymbol{T} \oplus \boldsymbol{C}\|, e = \|\ominus \boldsymbol{C} \oplus \boldsymbol{R}\|, f = \|\ominus \boldsymbol{R} \oplus \boldsymbol{A}\|$.

Proof : Let $\boldsymbol{P}$ is a point of the gyrotriangle $\boldsymbol{ABC}$ that three perpendiculars meet. Then the gyrosegments

$$\ominus \boldsymbol{A} \oplus \boldsymbol{P}, \quad \ominus \boldsymbol{B} \oplus \boldsymbol{P}, \quad \ominus \boldsymbol{C} \oplus \boldsymbol{P}, \quad \ominus \boldsymbol{S} \oplus \boldsymbol{P}, \quad \ominus \boldsymbol{T} \oplus \boldsymbol{P}, \quad \ominus \boldsymbol{R} \oplus \boldsymbol{P}$$

split the gyrotriangle $\boldsymbol{ABC}$ rinto six right gyrotriangles. Hence we can apply Theorem 3.5. to these gyrotrianles one by one. For simplicity, let

$$x = \|\ominus \boldsymbol{A} \oplus \boldsymbol{P}\|, y = \|\ominus \boldsymbol{B} \oplus \boldsymbol{P}\|, z = \|\ominus \boldsymbol{C} \oplus \boldsymbol{P}\|.$$

for the right gyrotriangles $\boldsymbol{APS}$ and $\boldsymbol{BPS}$, by the (3.5), we have

$$k^2 \oplus \left(\frac{\gamma_a}{\gamma_x}\right)^2 \otimes a^2 = x^2 \tag{4.1}$$

and

$$k^2 \oplus \left(\frac{\gamma_b}{\gamma_y}\right)^2 \otimes b^2 = y^2 \tag{4.2}$$

From (4.1) and (4.2), we obtain that

$$\left(\frac{\gamma_a}{\gamma_x}\right)^2 \otimes a^2 \ominus \left(\frac{\gamma_b}{\gamma_y}\right)^2 \otimes b^2 = x^2 \ominus y^2 \tag{4.3}$$

Similary, for the right gyrotriangles $\boldsymbol{BPT}$, $\boldsymbol{CPT}$ and $\boldsymbol{CPR}$, $\boldsymbol{APR}$ we obtain,

$$\left(\frac{\gamma_c}{\gamma_y}\right)^2 \otimes c^2 \ominus \left(\frac{\gamma_d}{\gamma_z}\right)^2 \otimes d^2 = y^2 \ominus z^2 \tag{4.4}$$

and

$$\left(\frac{\gamma_e}{\gamma_z}\right)^2 \otimes e^2 \ominus \left(\frac{\gamma_f}{\gamma_x}\right)^2 \otimes f^2 = z^2 \ominus x^2. \tag{4.5}$$

Then we have from (4.3),(4.4), (4.5), by Definition 2.4.(3)

$$\left(\frac{\gamma_a}{\gamma_x}\right)^2 \otimes a^2 \oplus \left(\frac{\gamma_c}{\gamma_y}\right)^2 \otimes c^2 \oplus \left(\frac{\gamma_e}{\gamma_z}\right)^2 \otimes e^2 = \left(\frac{\gamma_b}{\gamma_y}\right)^2 \otimes b^2 \oplus \left(\frac{\gamma_d}{\gamma_z}\right)^2 \otimes d^2 \oplus \left(\frac{\gamma_f}{\gamma_x}\right)^2 \otimes f^2, \tag{4.6}$$

and equivalently,

$$\left[(\gamma_y\gamma_z)^2\gamma_a^2\right] \otimes a^2 \oplus [(\gamma_x\gamma_z)^2\gamma_c^2] \otimes c^2 \oplus \left[(\gamma_x\gamma_y)^2\gamma_e^2\right] \otimes e^2 = [(\gamma_x\gamma_z)^2\gamma_b^2] \otimes b^2 \oplus \left[(\gamma_x\gamma_y)^2\gamma_d^2\right] \otimes d^2 \oplus \left[(\gamma_y\gamma_z)^2\gamma_f^2\right] \otimes f^2 \tag{4.7}$$

On the other hand, from Theorem 3.4., for these six right gyrotriangles we get

$$\gamma_x = \gamma_a\gamma_k = \gamma_t\gamma_f$$

$$\gamma_y = \gamma_k\gamma_b = \gamma_c\gamma_s$$

$$\gamma_z = \gamma_s\gamma_d = \gamma_t\gamma_e$$

These equations imply that

$$(\gamma_a\gamma_b\gamma_d)^2 \otimes a^2 \oplus (\gamma_a\gamma_c\gamma_d)^2 \otimes c^2 \oplus (\gamma_a\gamma_c\gamma_e)^2 \otimes e^2 = (\gamma_a\gamma_b\gamma_d)^2 \otimes b^2 \oplus (\gamma_a\gamma_c\gamma_d)^2 \otimes d^2 \oplus \left(\gamma_b\gamma_d\gamma_f\right)^2 \otimes f^2$$

Finally, we obtain

$$(\gamma_b\gamma_d)^2\left[\gamma_a^2 a^2 \ominus \gamma_f^2 f^2\right] \oplus (\gamma_a\gamma_d)^2\left[\gamma_c^2 c^2 \ominus \gamma_b^2 b^2\right] \oplus (\gamma_a\gamma_c)^2\left[\gamma_e^2 e^2 \ominus \gamma_d^2 d^2\right] = 0$$

5. Conclusion

The Einstein relativistic velocity model is a model of hyperbolic geometry. Many of theorems of Euclidean geometry are relatively similarto the Einstein relativistic velocity model, which is a model of hyperbolicgeometry. In Euclidean geometry, Carnot's theorem states that for a triangle $\boldsymbol{ABC}$ and the points $\boldsymbol{S}, \boldsymbol{T}, \boldsymbol{R}$ where located on the sides $\boldsymbol{BC}$, $\boldsymbol{AC}$ and $\boldsymbol{AB}$ respectively, then the perpendiculars to the sides of the triangle at the points $\boldsymbol{S}, \boldsymbol{T}$and $\boldsymbol{R}$ concurrent if and only if

$$a^2 - b^2 + c^2 - d^2 + e^2 - f^2 = 0. \tag{5.1}$$

where $a = |\boldsymbol{AS}|, b = |\boldsymbol{SB}|, c = |\boldsymbol{BT}|, d = |\boldsymbol{TC}|, e = |CR|, f = |\boldsymbol{RA}|$**.** In an Einstein gyrovector space $\mathbb{R}_s^n = (\mathbb{R}_s^n, \oplus, \otimes)$ for a gyrotriangle, $\boldsymbol{ABC}$ with vertices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}_s^n$, and sides $\boldsymbol{c} = \ominus \boldsymbol{A} \oplus \boldsymbol{B}, \boldsymbol{a} = \ominus \boldsymbol{B} \oplus \boldsymbol{C}$ and $\boldsymbol{b} = \ominus \boldsymbol{C} \oplus \boldsymbol{A}$. Carnot'stheorem (5.1) turns to

$$(\gamma_b \gamma_d)^2 \left[\gamma_a^2 a^2 \ominus \gamma_f^2 f^2\right] \oplus (\gamma_a \gamma_d)^2 \left[\gamma_c^2 c^2 \ominus \gamma_b^2 b^2\right] \oplus (\gamma_a \gamma_c)^2 \left[\gamma_e^2 e^2 \ominus \gamma_d^2 d^2\right] = 0 \qquad (5.2)$$

where $a = \|\ominus \boldsymbol{A} \oplus \boldsymbol{S}\|, b = \|\ominus \boldsymbol{S} \oplus \boldsymbol{B}\|, c = \|\ominus \boldsymbol{B} \oplus \boldsymbol{T}\|, d = \|\ominus \boldsymbol{T} \oplus \boldsymbol{C}\|, e = \|\ominus \boldsymbol{C} \oplus \boldsymbol{R}\|, f = \|\ominus \boldsymbol{R} \oplus \boldsymbol{A}\|$. In the Euclidean limit, of large s, $s \to \infty$, by Definition 2.2., gamma factor $\gamma_{\boldsymbol{u}}$ reduces to 1, and gyro equalty in 5.2 reduces to the

$$a^2 - b^2 + c^2 - d^2 + e^2 - f^2 = 0.$$

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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