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| RESEARCH ARTICLE

Optimizing Computational Techniques for Cauchy and Integral Equations Through Advanced Polynomial Methods

Summer W. Omar
Department of Mathematics, College of Education for Pure Sciences, Tikrit University, Tikrit, Iraq.
Corresponding Author: Summer W. Omar, E-mail: samar.watheq@tu.edu.iq

| ABSTRACT

This study introduces a numerical approach for solving Cauchy and integral equations using the Chebyshev pseudospectral method. The approach involves approximating the solution with an Nth-degree interpolating polynomial based on Chebyshev nodes, followed by problem discretization through a cell-averaging technique. The main properties of the Chebyshev pseudospectral method are discussed and explained to simplify the computation of Cauchy and integral equations into a system of algebraic equations. Several examples are presented to validate the method and to show how the method is computationally efficient, and to prove its effectiveness.

| KEYWORDS

Cauchy equation, inverse method, Chebyshev polynomial, iterative technique, applied mathematics

| ARTICLE INFORMATION

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1. Introduction

The class of solution procedures based on orthogonal polynomials is called spectral methods. They may be applied in different ways, including the tau, Galerkin, and collocation approaches, which have been proposed as potential strategies [1]. Among them, the most useful for computational reasons is the collocation method, also called the pseudospectral method. But in pseudospectral methods, the selected nodes must be the zeros of the derivatives of the classical orthogonal polynomials within the interval [-1,1], including endpoints. They are typically obtained from Legendre or Chebyshev polynomials.

Hallen [2] derived his now well-known integral equation in 1956 to provide an exact analysis of wave reflection of the current at the termination of a cylindrical tube-shaped antenna, although his early work on the subject was as early as 1938 [3]. With this equation, he was able to show that the current distribution on thin wire is sinusoidal in character and travels at approximately the speed of light. Hallen's integral equation is of the first kind of Fredholm integral equation.

The formula for a slender cylindrical antenna, which has a length of 1 and a radius much smaller than 1, is expressed as follows:

$$\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} K(x',y')I(y')dy' = \frac{j}{2\zeta_0} V\sin(\beta|x'|) + A\cos(\beta x') \tag{1}$$

There exist a pair of options for $K(x',y')$. These two kernels are commonly known as the precise and the simplified or diminished kernel. In the case of the simplified kernel, the integral equation does not yield a solution.

Despite this, the identical numerical technique is frequently utilized for both variations of the previously mentioned Equation (1). The estimated kernel employed in this research is described by.

$$\mathrm{K}(\mathrm{x}',\mathrm{y}') = \frac{1}{4\pi}\frac{\mathrm{e}^{-\mathrm{i}\beta\sqrt{(\mathrm{x}'-\mathrm{y}')^2+\mathrm{a}^2}}}{\sqrt{(\mathrm{x}'-\mathrm{y}')^2+\mathrm{a}^2}} \tag{2}$$

The parameters mentioned in Equations (1)–(2) are explained in detail in [4], such that the voltage parameter V does not exceed realistic values in any electrical scenario, considering a steady-state case as an example [4].

The other associated studies refer to initial boundary value problems of the subdiffusion type with spectral Galerkin methods in time and finite element [5] or finite difference [6] in space. They develop effective approaches for addressing singularities with significant improvement in the rate of convergence.

One of the most significant advantages of wavelets is their ability to remove or diminish singularities, as described in [7]. Wavelet bases have been extensively used for alternative representations of differential and integral operators [8], permitting more effective analysis of the governing equations. Wavelet bases have great advantages in numerical analysis, particularly in adaptive discontinuous Galerkin schemes [5], collocation schemes [9], and related schemes. As a case in point, Researchers in. [10] employed the Caputo fractional derivative and the Legendre wavelet method for solving differential equations and transformed the problem into solving algebraic equations and reducing it to finding unknown coefficients. Similarly, in [11], researchers integrated spectral schemes into finite element schemes with high-degree piecewise polynomial basis functions. This technique enhances accuracy with either increasing the spectral order or mesh refinement (hp method). hp-mesh schemes have quicker convergence with fewer degrees of freedom than conventional finite element schemes. hp methodology has limitations for complicated geometries.

Fractional calculus and fractional differential equations have wide-ranging applications in numerous scientific and engineering fields including material science of mechanical engineering, anomalous diffusion, robotics and control, signal processing, systems identification, friction modeling, wave propagation, turbulence, and fractal media seepage [12-14]. There are numerous equivalent definitions of fractional derivatives such as the Grüwald-Letnikov, Riemann-Liouville, and Caputo fractional derivatives [15].

2. Chebyshev Pseudospectral Method

The Chebyshev pseudospectral technique represents a specific instance within a broader category of spectral techniques. The fundamental approach of these techniques consists of two key phases: the first involves selecting a finite-dimensional space, typically a polynomial space, from which an approximation to the solution of a differential equation is derived. The second phase requires the selection of a projection operator that enforces the differential equation within the finite-dimensional space. A significant characteristic of spectral techniques is that the foundational polynomial space is composed of orthogonal polynomials that are globally differentiable without limits. Notable examples of these orthogonal polynomials include Legendre and Chebyshev polynomials, which maintain orthogonality over the interval $[-1,1]$, with respect to a suitable weight function $w(x) = 1$ for Legendre polynomials and $w(x) = (1-x^2)^{(-1/2)}$ for Chebyshev polynomials).

Let $\mathrm{P_N}$ represent the set of algebraic polynomials with a maximum degree of N. and let $T_m(x), m \gg 0, -1 \leq x \leq 1$, Identify the orthogonal set of Chebyshev polynomials of the first kind within this context, considering the associated weight function

$$w(x) = (1-x^2)^{(-1/2)}.$$

We opt for the points on the grid (for interpolation) to be.

$$x_j = \cos\left(\frac{j\pi}{N}\right),\ j = 0,1,\cdots,N \tag{3}$$

Chebyshev polynomials of order N. $\mathrm{T_N}(\mathrm{x}), \mathrm{x} \in [-1,1]$. These points are $x_\mathrm{N} = -1 < x_\mathrm{N} - 1 < \cdots < x_1 < x_0 = 1$, also views as the zeros of $(1-x^2)T_N(x)$, where $T_N(x) = dT_{N(x)}/dx$.

To create the interpolation for a given function f(x) at the point $\mathrm{x} \in [-1,1]$ and $\mathrm{k} = 0,1,\cdots,\mathrm{N}$, We establish the subsequent Lagrange polynomials.

$$\phi_k(x) = \frac{(-1)^{k+1}(1-x^2)T(x)}{c_k N^2 (x - x_k)} = \frac{2}{c_k}\sum_{j=0}^{N} \frac{T_j(x_k)T_j(x)}{c_j} \tag{4}$$

Which $c_0 = c_N = 2$ and $c_j = 1, j = 1,2,\cdots, N-1$. It is readily verified that

$$\phi_k(x_j) = \delta_{jk} \tag{5}$$

Associated with the $N+1$ Chebyshev points, which are specific grid locations, correspond to a distinct polynomial of degree N that serves as a projection operator $I_N f(x)$.

$$I_N f(x) = \sum_{j=0}^{N} \phi_j(x) f(x_j) \tag{6}$$

In a manner that $I_N f(x_k) = f(x_k), k = 0,1,\cdots, N$. On the other hand, the polynomial used for interpolation $I_N f(x)$. The chosen expression can be represented through the series expansion of traditional Chebyshev polynomials.

$$I_N f(x) = \sum_{j=0}^{N} T_j(x)\hat{F}(x_j) \tag{7}$$

where

$$\hat{F}(x_j) = \frac{2}{Nc_j}\sum_{r=0}^{N} \frac{T_r(x_j) f(x_r)}{c_r} \tag{8}$$

It is widely recognized that operators related to spectral projections, for instance, I_N, chosen points according to Chebyshev's criteria. x_j In comparison to those that rely on similar grid points, the subjects in question exhibit better behavior. Evidently, IN is a linear projection operator on $C[-1,1]$, the banach space of continuous, realvalued function on $[-1,1]$.

Here, we will employ the Chebyshev integration method based on cell-averaging. This principle indicates that there is a certain existence of an $N \times (N+1)$ matrix $R_{jk}, 1 \le j \le N, 0 \le k \le N$ such that for all $f \in C^1[-1,1]$, $r > 0$, we have

$$\begin{aligned} \int_{-1}^{1} f(x)dx &= \sum_{j=1}^{M} \int_{x_j}^{x_{j-1}} f(x)dx = \sum_{j=1}^{N} (x_{j-1} - x_i)\hat{f}_{j-\frac{1}{2}} \\ &= \sum_{j=1}^{N} (x_{j-1} - x_i) \sum_{k=0}^{N} R_{jk} f(x_k) \end{aligned} \tag{9}$$

Now let that

$$\mathrm{w_k} = \sum_{\mathrm{j=1}}^{\mathrm{N}} (x_{\mathrm{j-1}} - x_{\mathrm{j}})\mathrm{R_{jk}} \tag{10}$$

then the Equation (9) can be written as

$$\int_{-1}^{1} f(x)dx = \sum_{k=0}^{K} w_k f(x_k) \tag{11}$$

Where the cell-averages

$$\hat{\mathrm{f}}_{\frac{1}{2}}, \hat{\mathrm{f}}_{\frac{3}{2}}, \cdots, \hat{\mathrm{f}}_{\frac{2\mathrm{N}-1}{2}}$$

are related to $f(x_0), f(x_1), \cdots, f(x_N)$ throught the matrix $R_{ik,}$ $1 \le j \le N, 0 \le k \le N$. The entries of the cell-averaging matrix $R_{jk,}$ $1 \le j \le N, 0 \le k \le N$ are given by

$$\mathrm{R_{jk}} = \mathrm{g_k}\left(x_{\mathrm{j}-\frac{1}{2}}\right) \tag{12}$$

where

$$x_{j-\frac{1}{2}} = \cos\left(\frac{\left(j-\frac{1}{2}\right)\pi}{N}\right) \tag{13}$$

and also we have that

$$g_{\mathrm{k}}(\mathrm{x}) = \frac{1}{\mathrm{Nc_k}}\left[1 + \sigma_1 T_1(x_{\mathrm{k}})U_1(x) + \sum_{\mathrm{r=2}}^{\mathrm{N}} \frac{T_{\mathrm{r}}(\mathrm{x})[\sigma_{\mathrm{r}}\mathrm{U_r}(\mathrm{x}) - \sigma_{\mathrm{r-2}}\mathrm{U_{r-2}}(\mathrm{x})]}{\mathrm{c_r}}\right] \tag{14}$$

Which

$$\sigma_{\mathrm{r}} = \frac{\sin\left(\frac{r+1}{2N}\pi\right)}{(r+1)\sin\left(\frac{\pi}{2N}\right)}, \qquad U_r(x) = \frac{1}{r+1}T_{r+1}(x) \tag{15}$$

3. Numerical Approximation

First, we introduce the transformations $x' = \ell/2x$ and $y' = \ell/2y$. The proposed integral equation and the initial point $I(-\ell/2) = I(\ell/2) = 0$ can be represented as

$$\frac{\ell}{2}\int_{-1}^{1} K\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) I(y)dy = f(x), \ -1 < x < 1 \tag{16}$$

and

$$\mathrm{I}(-1) = \mathrm{I}(1) = 0 \tag{17}$$

where

$$f(x) = \frac{j}{2\zeta_0} V \sin\left(\beta\left|\frac{\ell}{2}x\right|\right) + A\cos\left(\beta\frac{\ell}{2}x\right), \qquad -1 < x < 1$$

When $x = y$, the kernel in Equation (16) exhibits a pronounced peak, especially when the value of 'a' is small. Consequently, from a computational perspective, it would be beneficial to separate and extract the singularity from the kernel. This can be achieved by expressing it in a different form such that $\mathrm{K}(\ell/2\mathrm{x}, \ell/2\mathrm{y})$ as

$$K\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) = K_n\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) + K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) \tag{18}$$

Where $\mathrm{K_n}(\ell/2\mathrm{x}, \ell/2\mathrm{y})$ and $\mathrm{K}_s(\ell/2\mathrm{x}, \ell/2\mathrm{y})$ denote the nonsingular and singular parts of kernel K, respectively and are given as:

$$K_n\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) = \frac{1}{4\pi}\frac{e^{-j\beta\sqrt{\left(\frac{\ell}{2}x - \frac{\ell}{2}y\right)^2 + a^2}}}{\sqrt{\left(\frac{\ell}{2}x - \frac{\ell}{2}y\right)^2 + a^2}} \tag{19}$$

$$K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) = \frac{1}{4\pi}\frac{1}{\sqrt{\left(\frac{1}{2}x - \frac{1}{2}y\right)^2 + a^2}} \tag{20}$$

Utilizing Equation (18) allows us to reformulate Equation (16).

$$\frac{\ell}{2}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) \mathrm{I}(y)dy + \frac{\ell}{2}\int_{-1}^{1} K_n\left(\frac{\ell}{2}x, \frac{\ell}{2}y\right) I(y)dy = f(x), -1 < x < 1 \tag{21}$$

The function being integrated in the initial Equation (21) is stable and can therefore be computed using numerical methods. Conversely, the function in the second term of Equation (21) has a singular point and will be computed in the manner described below. Let

$$\frac{\ell}{2}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)I(y)dy = S_1(x)+S_2(x) \tag{22}$$

where

$$S_1(x)=\frac{\ell}{2}\int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)\big(I(y)-I(x)\big)I(y)dy \tag{23}$$

and

$$S_2(x)=I(x)\int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)dy \tag{24}$$

The function being integrated in Equation (23) exhibits good behavior, and the integral presented in Equation (24) is computable.

$$H(x)=\int_{-1}^{1} K_s\left(\frac{\ell}{2}x,\frac{\ell}{2}y\right)\mathrm{d}y=\frac{1}{4\pi l}\ln\left[\frac{\sqrt{(lx-l)^2+4a^2}+lx-l}{\sqrt{(lx-l)^2+4a^2}-lx-l}\right] \tag{25}$$

In view of Equations (18) the integral term can be expressed as

$$\begin{aligned}\frac{\ell}{2}\int_{-1}^{1} \mathrm{K_n}\left(\frac{\ell}{2}\mathrm{x},\frac{\ell}{2}\mathrm{y}\right)I(y)dy+\frac{\ell}{2}\int_{-1}^{1} \mathrm{K_s}\left(\frac{\ell}{2}\mathrm{x},\frac{\ell}{2}\mathrm{y}\right)\big(I(y)-I(x)\big)dy\\+\frac{\ell}{2}I(x)H(x)=f(x),\qquad -1<x<1\end{aligned} \tag{26}$$

We will now break down Equation (24) utilizing the Chebyshev pseudospectral technique. In this approach, Equation (6) is employed for approximation.

$$I(y)=\sum_{j=0}^{N}\phi_j(x)I(y_j) \tag{27}$$

Where y_j and $\phi_j(y)$ for $0\le j\le N$, respectively are given Equations (3) and (4). A point-matching scheme is established by substituting Equations (25) into (24) and evaluating the result at the points x_k for $0\le k\le N$ given in Equation (3). This provides us with

$$\begin{aligned}&\frac{\ell}{2}\sum_{\mathrm{j}=0}^{N}\int_{-1}^{1} \mathrm{K_n}\left(\frac{\ell}{2}\mathrm{x_k},\frac{\ell}{2}\mathrm{y}\right)\phi_\mathrm{j}(\mathrm{y})\mathrm{I}(\mathrm{y_j})\mathrm{dy}+\\&\frac{\ell}{2}\sum_{\mathrm{j}=0}^{\mathrm{N}}\int_{-1}^{1} \mathrm{K_s}\left(\frac{\ell}{2}\mathrm{x_k},\frac{\ell}{2}\mathrm{y}\right)\big(\phi_\mathrm{j}(\mathrm{y})\mathrm{I}(\mathrm{y_j})-\mathrm{I}(\mathrm{x_k})\big)\\&\mathrm{dy}+\frac{\ell}{2}\mathrm{I}(\mathrm{x_k})\mathrm{H}(\mathrm{x_k})=\mathrm{f}(\mathrm{x_k}),\mathrm{k}=0,1,\cdots,\mathrm{N}\end{aligned} \tag{28}$$

Where we used $I(x_N)=0$, $I(x_j)=I_j$ for $0\le j\le N$. Furthermore, since the integrals in Equation (28) are well behaved, by using cell-averaging approximation of integrals in Equation (11), we approximate the Equation (28) as follows

$$\frac{\ell}{2}\sum_{j=0}^{N} I_j w_j K_n\left(\frac{\ell}{2}x_k,\frac{\ell}{2}y_i\right)+\frac{\ell}{2}\sum_{j=0}^{N}\left(I_j-I_k\right)w_i K_s\left(\frac{\ell}{2}x_k,\frac{\ell}{2}y_j\right)+\frac{\ell}{2}I(x_k)H(x_k)=f(x_k), k=0,1,\cdots,N \tag{29}$$

By solving the system of linear Equation (27), we can find I_j for $j=0,1,\cdots,N-1$.

4. Results and Discussion

This research presents a numerical case to demonstrate the effectiveness and advantages of this method. This instance provides information for two chosen wire lengths as examples of integral equations and one metal piece as an example of Cauchy equation, encompassing unique scenarios of practical significance., e.g $I=\lambda$, $I=\lambda/2$, and $I=\lambda/4$ respectively. The scales of electrical flows I(y) are shown for $N=5, 10$ and 20 in Figures 1 to 3, respectively. As illustrated in Figures 1 to 3, the solution approaches convergence quickly as the values of N are increased.

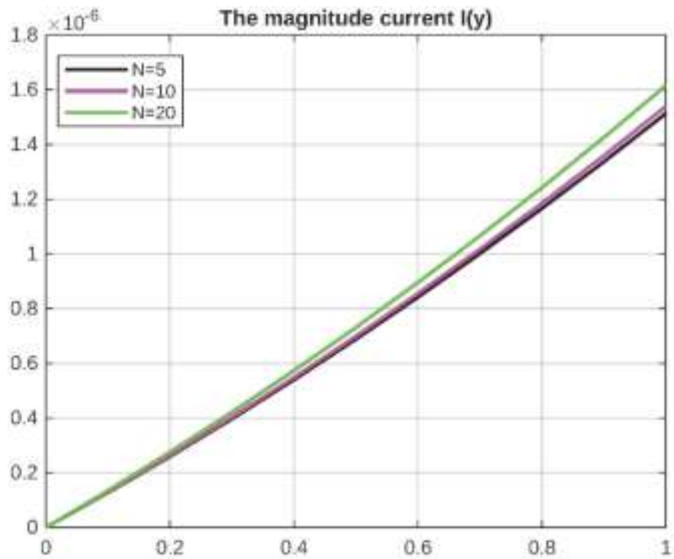

Figure 1: The scales of electrical flows $I(y)$ for $I=\lambda$

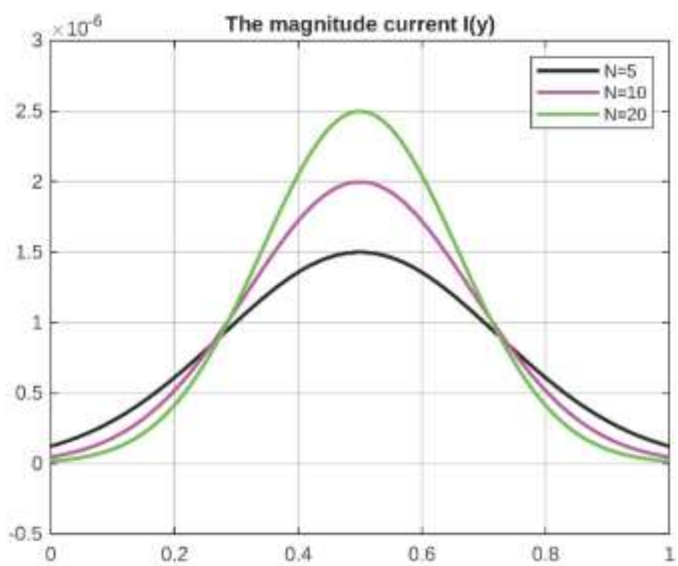

Figure 2: The scales of electrical flows $I(y)$ for $I=\lambda/2$

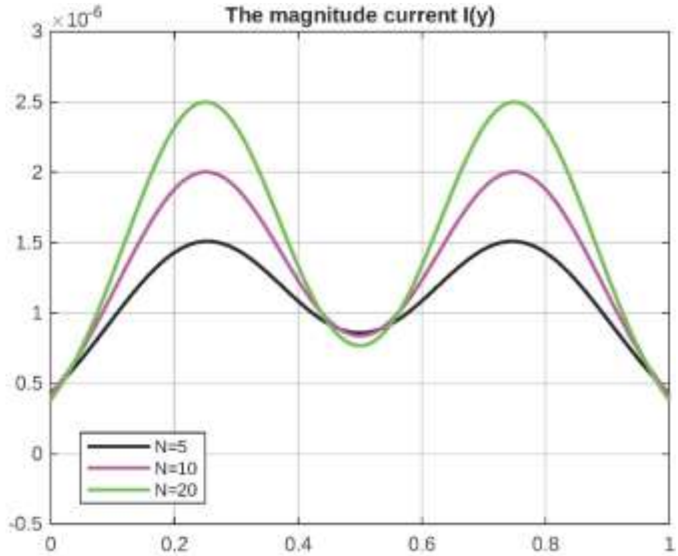

Figure 3: The scales of electrical flows $\mathrm{I}(\mathrm{y})$ for $\mathrm{I} = \lambda/4$

5. Conclusion

The paper studied a novel approach for dealing with integral and Cauchy equations using Chebyshev polynomial as a tool for the derivation of the approach. Three examples were presented and the results showed that the proposed method is effective. For the validation of the proposed method, a variety of magnitude currents I(y) were taken, and in each one the convergence is verified. This study can be extended in future to deal with fractional order equations and to be applied on a variety of applications in many different fields.

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ORCID: https://orcid.org/0009-0006-5243-3221

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